VECTOR VALUED MEASURES AND FUNCTIONS ON A LOCALLY COMPACT GROUP

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TO

MY MOTHER

CERTIFICATE

This is to certify that the work embodied in the thesis "Vector valued measures and functions on a locally compact group" by Malayananda Dutta, has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

November 30, 1978

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CONTENTS

CHAPTER I

INTRODUCTION

In this thesis, we study certain aspects of Harmonic Analysis of vector-valued functions. The objects of study in Abstract Harmonic Analysis ([13],[14],[25] and [34]), are the spaces of complex-valued, integrable functions and measures on a locally compact group. We propose to derive some results in the more generalised situation, where these functions and measures take values in an arbitrary Banach space or a Banach algebra. The theory of vector measures and integration on arbitrary spaces, has been developed in [15], and more completely in [1]. Using this theory, Hausner ([10], [11] and [12]) and Johnson [18] studied the vector-valued group algebra $L^1(G,A)$ of A-valued (Bochner) integrable functions on G, where G is a locally compact abelian group, and A is a commutative Banach algebra. We study this algebra in more detail. We also prove some results on $L^p(G,X)$ and $M(G,X)$, the space of X-valued, regular, Borel measures of bounded variation on G, where G is any locally compact group, and X is any Banach space.

In the second chapter, we collect the basic definitions and known results that we shall need subsequently. First, we

give a systematic review of the theory of vector measures and integration, with special emphasis on locally compact Hausdorff spaces. After that, we give the definitions and results on convolutions of vector-valued functions and measures on a locally compact group. Finally, we discuss the module tensor products and their relation to multipliers. We avoid unnecessary generalities and present the results in the form in which we need them. In some cases, (e.g. the Riesz Representation Theorem for the vector-valued case) we give our own proofs.

In the third chapter, we prove some results on the vector-valued group algebra $L^1(G,A)$, where G is a locally compact abelian group, and A is a commutative, semisimple Banach algebra. The results connect some properties of $L^1(G,A)$ with the corresponding properties of A. We prove that (i) $L^1(G,A)$ has a bounded approximate identity if and only if A has a bounded approximate identity, and that (ii) $L^1(G,A)$ is Tauberian if and only if A is Tauberian. The main result of the third chapter is that $L^1(G,A)$ is isometrically isomorphic to a group algebra if and only if A is isometrically isomorphic to a group algebra. It is already known (Theorem 3.2 of [18]) that if A is isometrically isomorphic to $L^1(H)$, for some locally compact abelian group H, then $L^1(G,A)$ is isometrically isomorphic to $L^1(G \times H)$. The question is whether

the converse is true. In [30], Rieffel gives a characterisation of those commutative Banach algebras which are isometrically isomorphic to the group algebra of a locally compact abelian group. We use this characterisation to prove our result. Thus, apart from its intrinsic interest, our result gives an interesting application of Rieffel's results. Rieffel also characterised those Banach algebras which are isometrically isomorphic to $M(H)$, the algebra of regular Borel measures of bounded variation on a locally compact abelian group H. We use this to prove that if the projective tensor product $M(G) \otimes_\gamma A$ is isometrically isomorphic to $M(H)$, for some locally compact abelian group H, then A is isometrically isomorphic to $M(G_1)$ for some locally compact abelian group G_1. By Theorem 4.2 of [22], there is an isometric isomorphism of $M(G) \otimes_\gamma A$ into $M(G,A)$ and, under certain conditions, this isomorphism is onto. Thus, our result on $M(G) \otimes_\gamma A$ gives a partial converse of the following result which we prove earlier. If A is isometrically isomorphic to $M(H)$, for some locally compact abelian group H, then $M(G,A)$ is isometrically isomorphic to $M(G \times H)$. We do not know whether the full converse of this result is true. It will be interesting to investigate this.

The main result of the fourth chapter, is the characterisation of the continuously translating vector-valued measures

on a locally compact group. Let G be a locally compact group, and X a Banach space. For $\mu \in M(G,X)$, and $s \in G$, μ_s and $_s\mu$ denote, respectively, the right and the left translate of μ by s. We prove that if $\mu \in M(G,X)$, such that either $s \to \mu_s$ or $s \to {}_s\mu$ is continuous, then $\mu \in L^1(G,X)$. The scalar-valued version of this result is well-known. In this case, the result is proved in § 19.27 of [13] by proving the absolute continuity of μ with respect to the Haar measure. The same proof can be modified in the vector-valued case, to prove that μ is absolutely continuous with respect to the Haar measure. However, this is not enough to conclude that $\mu \in L^1(G,X)$ because of the absence of the Radon Nikodym theorem in the vector-valued case. In [4], Gaudry has proved the scalar-valued case of this result by using Weil's criterion [42] of relatively compact subsets of $L^p(G)$. We first generalise Weil's criterion to the vector-valued case, and then use Gaudry's idea to prove our result.

In the fifth chapter, we determine the multipliers of $L^1(G,A)$, where G is a locally compact abelian group, and A is a commutative, semisimple Banach algebra with an identity of norm one. To do this, we first characterise the invariant operators (i.e. the operators which commute with translations) from $L^1(G)$ into $L^1(G,X)$, for any Banach space X. We prove that the space of invariant operators

from $L^1(G)$ into $L^1(G,X)$ is isometrically isomorphic to $M(G,X)$. We also prove that the space of invariant operators of $L^1(G,X)$, is isometrically isomorphic to $\mathcal{L}(X,M(G,X))$, the space of bounded linear operators from X into $M(G,X)$. Using these results, we prove that the space of multipliers of $L^1(G,A)$, is isometrically isomorphic to $M(G,A)$, generalising a well-known result (see Theorem 0.1.1 of [23]) in the scalar-valued case.

In the sixth chapter, we prove some general results on multipliers of certain commutative, semisimple Banach algebras. Let T be a multiplier of a commutative, semisimple Banach algebra A, such that T is the product of an idempotent and an invertible multiplier. Then it is easy to see that $T^2(A)$ is closed. In the sixth chapter, among other things, we prove the converse of this for any regular, commutative, semisimple, Tauberian Banach algebra satisfying a special condition. These results arose in connection with [7], where the special cases of these results for $L^1(G)$, were proved. The proofs immediately generalise to all Banach algebras satisfying the special hypothesis mentioned in §4 of [7]. In an attempt to prove such results for the vector-valued group algebras, we managed to prove some general results applicable to a class of algebras larger than those mentioned in §4 of [7]. For example, our results are applicable to the following algebra which does not

satisfy the hypothesis of §4 of [7]. Let T be the circle group, and let $A = \{ f \in L^1(T): \sum_{n=-\infty}^{\infty} |n\hat{f}(n)| < \infty \}$ with $||f||_A = ||f||_{L^1(T)} + \sum_{n=-\infty}^{\infty} |n\hat{f}(n)|$. With the help of our general results, we are able to prove the corresponding results for some vector-valued group algebras. We show that our results also apply to all Segal algebras, and this leads to an interesting proof of the already known fact (see Proposition 2.2 of [21]) that, for a Segal algebra on a noncompact group, there is no nontrivial compact multiplier. Finally, for $\mu \in M(G)$, we prove that the existence of a bounded projection (not necessarily a multiplier) from $L^1(G)$ onto $\mu * L^1(G)$, implies that $\mu * L^1(G)$ is closed if and only if μ is the convolve of an idempotent and an invertible measure. This is a small step towards proving Glicksberg's conjecture [7], that $\mu * L^1(G)$ is closed if and only if μ is the convolve of an idempotent and an invertible measure. This conjecture has been recently proved by B. Host and F. Parreau [16], who have an entirely different approach.

CHAPTER II

VECTOR VALUED FUNCTIONS AND MEASURES

2.1 <u>Vector-valued set functions</u>: Let S be an arbitrary set, and Σ be a σ-ring of subsets of S. We define $T(\Sigma) = \{E \subset S : E \cap F \in \Sigma \text{ for any } F \in \Sigma\}$. It is easy to see that $\Sigma \subset T(\Sigma)$, and that $T(\Sigma)$ is a σ-algebra of subsets of S. Moreover, if Σ is a σ-algebra, then $T(\Sigma) = \Sigma$. Let X be an arbitrary Banach space and let μ be a set function from Σ into X. The total variation $V(\mu)$ of μ, is a non-negative, extended real-valued set function, defined for any $E \subset S$ as follows, $V(\mu)(E) = \sup\{\sum_{i=1}^{n} ||\mu(E_i)|| : E_i\text{'s disjoint}, E_i \subset E \text{ for } 1 \leq i \leq n\}$, the supremum being taken for all possible choices of E_i's. The proofs of the following are trivial (see [1]).

<u>Proposition 2.1</u>: $V(\mu)$ is a monotonic set function, i.e. $E_1 \subset E_2$ implies $V(\mu)(E_1) \leq V(\mu)(E_2)$.

<u>Proposition 2.2</u>: If μ is finitely additive, then $V(\mu)$ is finitely additive on $T(\Sigma)$.

<u>Proposition 2.3</u>: If μ is countably additive, then $V(\mu)$ is countably additive on $T(\Sigma)$.

An X-valued measure on (S,Σ) is nothing but a countably additive set function from Σ into X. For any

X-valued measure μ, $||\mu(E)||$ for $E \in \Sigma$, is bounded. μ is said to be of bounded variation if $V(\mu)$ is finite. For two vector-valued measures μ and ν, μ is said to be absolutely continuous with respect to ν, if $V(\mu)$ is absolutely continuous with respect to $V(\nu)$. The space $\tilde{M}(S,\Sigma,X)$ of X-valued measures of bounded variation on S, forms a Banach space under the norm $||\mu||_V = V(\mu)(S)$.

2.2 Measurable functions: For $x \in X$, and a scalar-valued function f on S, the X-valued function $F = fx$ is defined by $F(s) = f(s)x$ for all $s \in S$. An X-valued function F on S, is called a simple function if it is of the form $F = \sum_{i=1}^{n} \chi_{E_i} x_i$, where $x_i \in X$, and χ_{E_i} is the characteristic function of E_i with $E_i \in \Sigma$, for $1 \leq i \leq n$. (Hereafter, χ_E will always denote the characteristic function of E.) F is called countably-valued if $F = \sum_{i=1}^{\infty} \chi_{E_i} x_i$, where each $x_i \in X$, and $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets belonging to Σ. Let μ be a vector measure on (S,Σ). An X-valued function F on S, is called μ-measurable if there exists a sequence of countably-valued functions converging to F a.e. $[V(\mu)]$. F is called weakly μ-measurable if the scalar-valued function $\phi \circ F$ is μ-measurable for every $\phi \in X^*$, the dual of X. We note that for scalar-valued functions, any measurable function in the sense of Halmos [9], is μ-measurable for any measure μ.

Also, any μ-measurable, scalar-valued function is equal a.e. $[V(\mu)]$, to a measurable function in the sense of Halmos. The following proposition is well-known (Theorem 3.5.3 of [15]), and provides us with a convenient way to test the measurability of vector-valued functions.

Proposition 2.4: Let μ be a vector measure on (S,Σ). An X-valued function F on S, is μ-measurable if and only if F is weakly μ-measurable, and there exists a set $E \subseteq S$ with $V(\mu)(E) = 0$, such that $F(S-E)$ is separable.

For an X-valued function F on S, $|F|$ will denote the function defined by $|F|(s) = ||F(s)||$ for all $s \in S$.

2.3 Integration (Bochner): Let X,Y,Z be Banach spaces with a bilinear map $(x,y) \to x.y$ from $X \times Y$ into Z, such that $||x.y||_Z \leq ||x||_X \, ||y||_Y$. The triple (X,Y,Z) will be said to form a Bilinear system of Banach spaces (see [17]). Let μ be an Y-valued measure on (S,Σ), and let F be an X-valued, simple or countably-valued function on S, of the form $F = \sum_i \chi_{E_i} x_i$. F is called μ-integrable if $\sum_i V(\mu)(E_i) \, ||x_i||_X$ is finite, and in this case, we define $\int F d\mu = \sum_i x_i.\mu(E_i)$. $\int F d\mu$ is well-defined as a member of Z, since $\sum_i ||x_i.\mu(E_i)||_Z \leq \sum_i ||x_i||_X \, ||\mu(E_i)||_Y \leq \sum_i ||x_i||_X \, V(\mu)(E_i) < \infty$. A μ-measurable, X-valued function F on S, is called μ-integrable, if there is a sequence $\{F_n\}_{n=1}^{\infty}$ of X-valued, μ-integrable, countably-valued

functions converging to F a.e. $[V(\mu)]$, with $\lim_{m,n\to\infty} \int ||F_m - F_n||\, dV(\mu) = 0$. In this case, $\{\int F_n d\mu\}_{n=1}^{\infty}$ is a Cauchy sequence in Z, and it can be shown that $\lim_{n\to\infty} \int F_n\, d\mu$ depends only on F, and not on the choice of the sequence $\{F_n\}$. We define $\int F d\mu = \lim_{n\to\infty} \int F_n\, d\mu$. If F is μ-integrable, and $E \in \Sigma$, then it is easy to see that $\chi_E F$ is μ-integrable, and we define $\int_E F d\mu = \int \chi_E F\, d\mu$. We note that this integral of vector-valued functions with respect to vector-valued measures, satisfies all the typical properties of an integral. For example, it is easy to prove that $||\int F\, d\mu\,|| \leq \int |F|\, dV(\mu)$. Also, if F is μ-integrable, then $E \to \int_E F\, d\mu$ gives a Z-valued measure on (S,Σ).

We shall denote the field of complex numbers by C. For any Banach space X, the triple (X,C,X) forms a Bilinear system in a natural way, if we take $x.\alpha = \alpha x$ for $\alpha \in C$ and $x \in X$. This leads us to the theory of integration of vector-valued functions with respect to scalar-valued measures, as developed in §3.5 - §3.8 in [15]. In this case, we get the following.

<u>Proposition 2.5:</u> Let μ be a scalar measure, and F an X-valued, μ-integrable function on S. Let T be a bounded linear map from X into another Banach space Y. Then $T \circ F$ is an Y-valued, μ-integrable function, and

$$\int (T \circ F)\, d\mu = T\left(\int F\, d\mu \right).$$

The proof is easy (see Theorem 3.7.12 of [15], and its subsequent comments).

Similarly (C, X, X) forms a Bilinear system under $\alpha . x = \alpha x$, for $\alpha \in C$ and $x \in X$. This gives the theory of integration of scalar-valued functions with respect to vector-valued measures. In this case, we get

<u>Proposition 2.6</u>: Let μ be an X-valued measure, and f a scalar-valued, μ-integrable function on S. Let T be a bounded linear map from X into another Banach space Y. Then $T \circ \mu$ is an Y-valued measure, f is $(T \circ \mu)$ integrable, and

$$\int f\, d(T \circ \mu) = T\left(\int f d\mu \right)$$

The following proposition provides us with a convenient way to test the integrability of vector-valued functions with respect to vector-valued measures.

<u>Proposition 2.7</u>: Let μ be an Y-valued measure, and F an X-valued, μ-measurable function on S. Then the following are equivalent.

(a) F is μ-integrable.

(b) $|F|$ is μ-integrable.

(c) F is $V(\mu)$-integrable.

(d) $|F|$ is $V(\mu)$-integrable.

The proof is similar to that of Theorem 3.7.4 of [15].

2.4 Spaces of Integrable functions: Let λ be a fixed positive measure on (S,Σ). Two λ-measurable, X-valued functions on S are called λ-equivalent, if they are equal a.e.(λ). For $1 \leq p < \infty$, $L^p(S,\Sigma,\lambda,X)$ will denote the set of λ-equivalence classes of λ-measurable, X-valued functions, such that, if F is a representative of an equivalence class belonging to $L^p(S,\Sigma,\lambda,X)$, then $(\int |F|^p d\lambda)^{1/p} = |||F|||_p < \infty$. $L^p(S,\Sigma,\lambda,X)$ forms a Banach space with the norm $|||\cdot|||_p$. (In the scalar-valued case, the norm will be denoted by $||\cdot||_p$, instead of $|||\cdot|||_p$). As usual, by a function in $L^p(S,\Sigma,\lambda,X)$, we shall mean the corresponding equivalence class. If, in a particular context, the choice of Σ and λ are clear (e.g. for a locally compact group, the σ-ring of Borel sets and the left Haar measure), then these spaces will be denoted by $L^p(S,X)$.

If we take any $F \in L^1(S,\Sigma,\lambda,X)$, then from the definition and Proposition 2.7, it follows that F is λ-integrable. Hence, we can define the mapping $\mu_F: \Sigma \to X$, by $\mu_F(E) = \int_E F d\lambda$. It is easy to see that μ_F is an X-valued measure of bounded variation on S, and that $F \to \mu_F$ gives an isometric imbedding of $L^1(S,\Sigma,\lambda,X)$ in $\tilde{M}(S,\Sigma,X)$. When there is no scope of confusion (as in Chapter V), we shall use the same symbol for $F \in L^1(S,\Sigma,\lambda,X)$ and its canonical image in $\tilde{M}(S,\Sigma,X)$. Similarly if $\mu \in \tilde{M}(S,\Sigma,X)$, then

'' μ is in $L^1(S,\Sigma,\lambda,X)$'' will mean that μ is of the form μ_F for some $F \in L^1(S,\Sigma,\lambda,X)$.

All the scalar-valued function and measure spaces will be denoted by the corresponding symbol for the vector-valued case, with the index 'X' suppressed. Thus, $L^p(S,\Sigma,\lambda)$ will denote the usual scalar-valued L^p-spaces, $\bar{M}(S,\Sigma)$ will be the space of complex measures and so on.

2.5 <u>$L^1(S,\Sigma,\lambda,X)$ as a tensor product</u>: Let E and F be two Banach spaces. The algebraic tensor product (see §6 of [36] for definition) will be denoted by $E \otimes F$. Every element of $E \otimes F$ can be expressed as a finite sum $\sum_i \lambda_i (e_i \otimes f_i)$ (the sum over empty set being 0), with each $\lambda_i \in C$, $e_i \in E$ and $f_i \in F$. All equivalent expressions for the same element of $E \otimes F$ can be obtained from one such expression by the use of the following identities,

$$\begin{aligned} \lambda(e \otimes f) &= (\lambda e) \otimes f = e \otimes (\lambda f) \\ (e_1 + e_2) \otimes f &= e_1 \otimes f + e_2 \otimes f \\ e \otimes (f_1 + f_2) &= e \otimes f_1 + e \otimes f_2 . \end{aligned}$$

The greatest cross-norm (see [37]) on $E \otimes F$ is given by

$$||t||_\gamma = \inf \{ \sum_i ||e_i|| \, ||f_i|| : t = \sum_i e_i \otimes f_i \}$$

where the infimum is taken over all possible expressions of t. The projective tensor product of E and F (see [8]),

denoted by $E \otimes_\gamma F$, is nothing but the completion of $E \otimes F$ with the greatest cross-norm. Every element $t \in E \otimes_\gamma F$ can be expressed as $t = \sum_{i=1}^{\infty} e_i \otimes f_i$, with each $e_i \in E$, and $f_i \in F$, such that

$$\sum_{i=1}^{\infty} ||e_i||\ ||f_i|| < \infty .$$

The projective tensor product satisfies the following universal property with respect to bounded bilinear maps from $E \times F$, the cartesian product of E and F. Any bilinear map T from $E \times F$ into a Banach space H, such that $||T(e,f)|| \leq K\ ||e||\ ||f||$, gives rise to a bounded linear transformation T_γ, from $E \otimes_\gamma F$ into H, with $||T_\gamma|| \leq K$, such that $T_\gamma\ (e \otimes f) = T(e,f)$.

Let $f \in L^1(S,\Sigma,\lambda)$ and let $x \in X$. It is easy to see that the X-valued function $fx \in L^1(S,\Sigma,\lambda,X)$, and that $|||fx|||_1 = ||f||_1\ ||x||$. Hence $(f,x) \to fx$ gives a bounded bilinear map from $L^1(S,\Sigma,\lambda) \times X$ into $L^1(S,\Sigma,\lambda,X)$. The resulting bounded linear map, from $L^1(S,\Sigma,\lambda) \otimes_\gamma X$ into $L^1(S,\Sigma,\lambda,X)$, can be proved (see 6.5 of [36]) to be an isometric isomorphism between these two spaces. This implies, in particular, that functions of the form $\sum_{i=1}^{n} f_i x_i$, with each $f_i \in L^1(S,\Sigma,\lambda)$, and each $x_i \in X$, are dense in $L^1(S,\Sigma,\lambda,X)$.

2.6 Radon Nikodym Property: When $\mu \in \bar{M}(S,\Sigma,X)$ is of the form μ_F for some $F \in L^1(S,\Sigma,\lambda,X)$, then we shall say that μ has the derivative F with respect to λ. X is said to satisfy the Radon-Nikodym property (X has RNP), if any X-valued measure μ of bounded variation on an arbitrary measurable space (S,Σ), has a derivative with respect to $V(\mu)$. Any reflexive Banach space and any separable Banach space which is the dual of some other Banach space, have RNP (see [27] and [2]). An example of a separable Banach space which does not have RNP is $L^1[0,1]$ (see [41]). If X has RNP, then for any positive measure λ on (S,Σ), the canonical image of $L^1(S,\Sigma,\lambda,X)$ in $\bar{M}(S,\Sigma,X)$, is precisely the set of measures in $\bar{M}(S,\Sigma,X)$, which are absolutely continuous with respect to λ.

2.7 Locally compact Hausdorff space, Regularity: Let S be a locally compact Hausdorff space, and let Σ be the σ-ring of sets generated by the class of compact subsets of S. By 'Borel sets', we will mean the sets contained in Σ. It is easy to see that every open set belongs to $\Upsilon(\Sigma)$. (By 'Borel sets', some authors mean the elements of the σ-algebra generated by the open sets.) By a vector-valued Borel measure on S, we shall mean a vector measure on the σ-ring Σ of Borel subsets of S. Any scalar-valued continuous function vanishing at infinity, is Borel measurable in the sense of Halmos [9], and hence (see §2.2) is μ-measurable for any Borel measure μ.

Let μ be any X-valued set function on Σ. μ is called regular if for every $E \in \Sigma$, and for every $\varepsilon > 0$, there exists a compact set K and an open set U, with $K \subset E \subset U$, such that, for any $F \in \Sigma$ with $K \subset F \subset U$, we have $||\mu(E) - \mu(F)|| < \varepsilon$. A positive, extended real-valued, Borel measure μ is called regular if, in addition to the above condition, we have $\mu(K) < \infty$, for any compact set $K \subset S$. μ is called <u>inner-regular</u>, if for every $E \in \Sigma$, and for every $\varepsilon > 0$, there exists a compact set $K \subset E$, such that, for any $F \in \Sigma$ with $K \subset F \subset E$, we have $||\mu(E) - \mu(F)|| < \varepsilon$. Similarly, we can define outer-regularity. It is easy to see that if μ is finitely additive then μ is regular if and only if μ is both inner and outer-regular. The following proposition shows that for finitely additive set functions of bounded variation, the condition of regularity is much stronger than it appears.

<u>Proposition 2.8</u>: Let $\mu : \Sigma \to X$, be a finitely additive set function with finite variation $V(\mu)$. Then μ is regular if and only if $V(\mu)$ is regular.

<u>Proof</u>: The proof of the 'if' part is trivial, because of the fact that, $V(\mu)$ is finitely additive, and that $||\mu(E)|| \leq V(\mu)(E)$ for $E \in \Sigma$.

Conversely, if μ is regular, then by Proposition 19 of §15 of [1], $V(\mu)$ is inner regular. Regularity of $V(\mu)$

will now follow from Proposition 4 of §15 of [1], if we can prove that for every $E \in \Sigma$, there exists $E' \in \Sigma$, such that $E \subset$ Interior E'. But this is easily proved and our proof is complete.

From Proposition 2.8, it follows that an X-valued, finitely additive function of bounded variation on Σ, is regular, if and only if for every $E \in \Sigma$, and for every $\epsilon > 0$, there exists a compact set K and an open set U with $K \subset E \subset U$, such that $V(\mu)(U-K) < \epsilon$. Using this, the following is easy to prove (see the comments following Definition II.6 of [17]).

Proposition 2.9. A finitely additive, regular X-valued set function on Σ is countably additive.

The space of regular, X-valued Borel measures of bounded variation on S will be denoted by $M(S,X)$. $M(S,X)$ is a Banach space with the norm $||\mu||_V = V(\mu)(S)$. If λ is a positive, extended real-valued, regular, Borel measure, then any X-valued measure absolutely continuous with respect to λ, is also regular. Hence, in such a case, the canonical image of $L^1(S,\Sigma,\lambda,X)$ in $\bar{M}(S,\Sigma,X)$, is actually contained in $M(S,X)$.

2.8 $C_0(S,X)$ and its dual: Let S be a locally compact Hausdorff space. An X-valued function F on S, is said to have compact support, if $|F|$ has compact support. F will be said to be vanishing at infinity, if $|F|$ vanishes

at infinity. $C_c(S,X)$ will denote the space of X-valued continuous functions on S, with compact support. $C_0(S,X)$ will denote the space of X-valued continuous functions on S, vanishing at infinity. $C_0(S,X)$ with the norm $|||F|||_\infty = \sup_{s \in S} ||F(s)||$ is a Banach space and $C_c(S,X)$ is dense in $C_0(S,X)$. Let λ be a regular, positive, extended real-valued, Borel measure. It is easy to show that the functions of the form $\sum_{i=1}^{n} f_i x_i$, with each $f_i \in C_c(G)$, and each $x_i \in X$, are dense in $L^1(S,\Sigma,\lambda,X)$. In particular, $C_c(S,X)$ is dense in $L^1(S,\Sigma,\lambda,X)$. Moreover, the following is easy to prove.

<u>Proposition 2.10</u>: The functions of the form $\sum_{i=1}^{n} f_i x_i$, with $f_i \in C_0(S)$ and $x_i \in X$ for $1 \leq i \leq n$, are dense in $C_0(S,X)$.

In the scalar-valued case, by the Riesz Representation theorem, the dual of $C_0(S)$ is isometrically isomorphic to $M(S)$. The analogue of this in the vector-valued case, is the following.

<u>Theorem 2.1</u>. (Riesz Representation Theorem): Let $\mu \in M(S,X^*)$ where X^* is the dual of X. Then μ defines an element of the dual of $C_0(S,X)$, in the following way, $\langle \mu, F \rangle = \int F d\mu$, for every $F \in C_0(S,X)$. This correspondence gives an isometric isomorphism between $C_0(S,X)^*$ and $M(S,X^*)$.

This result is well-known, and follows from §19 of [1]. For compact S, this is also proved by I.Singer [38]

(see the proof of Theorem 5.2 of [22]). In the following, we give a simple proof of Theorem 2.1, based on the corresponding result in the scalar-valued case.

<u>Proof of Theorem 2.1</u>: Let $F \in C_0(S,X)$ and $\mu \in M(S,X^*)$. The triple (X,X^*,C) forms a Bilinear system and by §2.3, $\int F d\mu$ will be defined as a complex number, if F is μ-integrable. Now for any $\phi \in X^*$ and $F \in C_0(S,X)$, $\phi o F$ is a scalar-valued, continuous function vanishing at infinity and hence is μ-measurable. Thus, F is weakly μ-measurable. Also, the range of F is separable, since $F \in C_0(S,X)$. Hence, by Proposition 2.4, F is μ-measurable. Moreover, $\int |F| dV(\mu) \leq |||F|||_\infty ||\mu||_V < \infty$. Thus, by Proposition 2.7, F is μ-integrable, and $|\int F d\mu| \leq |||F|||_\infty ||\mu||_V$. Obviously, $F \to \langle\mu,F\rangle = \int F d\mu$, is linear and thus, each $\mu \in M(S,X^*)$, defines an element of $C_0(S,X)^*$.

Conversely, let $L \in C_0(S,X)^*$. For any $x \in X$, define $\ell^{(x)}: C_0(S) \to C$, by $\ell^{(x)}(f) = L(fx)$, for any $f \in C_0(S)$. $\ell^{(x)}$ is obviously linear. Also $|\ell^{(x)}(f)| = |L(fx)| \leq ||L||\ |||fx|||_\infty = ||L||\ ||x||\ ||f||_\infty$. Thus $\ell^{(x)} \in C_0(S)^*$ and $||\ell^{(x)}|| \leq ||L||\ ||x||$. By the Riesz Representation theorem for the scalar-valued case, there exists $\mu^{(x)} \in M(S)$, such that $\ell^{(x)}(f) = \int f d\mu^{(x)}$, for all $f \in C_0(S)$. Also $||\mu^{(x)}||_V = ||\ell^{(x)}|| \leq ||L||\ ||x||$. Now for $E \in \Sigma$, $|\mu^{(x)}(E)| \leq ||\mu^{(x)}||_V \leq ||L||\ ||x||$. Thus $x \to \mu^{(x)}(E)$

gives a bounded linear functional on X. We denote this element of X^* by $\mu_L(E)$, and we have $||\mu_L(E)|| \leq ||L||$. From the definitions, it is clear that μ_L is finitely additive on Σ. We shall now show that μ_L is regular and of bounded variation, and then by Proposition 2.9, we will be able to conclude that $\mu_L \in M(S,X^*)$.

Let $C_0(S)^+ = \{f \in C_0(S): f \geq 0\}$. Define a nonnegative function L_V on $C_0(S)^+$, by $L_V(f) = \sup\{|L(F)|: F \in C_0(S,X), |F| \leq f\}$. It is easy to prove that L_V is additive on $C_0(S)^+$, and $L_V(\alpha f) = \alpha L_V(f)$ for any $f \in C_0(S)^+$ and $\alpha \geq 0$. Moreover, for $F \in C_0(S,X)$ with $|F| \leq f$, $|L(F)| \leq ||L||\ |||F|||_\infty \leq ||L||\ ||f||_\infty$. Therefore, $L_V(f) \leq ||L||\ ||f||_\infty$. So L_V can be extended to the whole of $C_0(S)$ to give a positive, bounded linear functional on $C_0(S)$, with $||L_V|| \leq ||L||$. Now, if $F \in C_0(S,X)$ with $|||F|||_\infty \leq 1$, then $|L(F)| \leq L_V(|F|) \leq ||L_V||$. Therefore, $||L|| \leq ||L_V||$, and hence $||L_V|| = ||L||$. Let λ be the positive measure in $M(S)$, which represents L_V. Then $L_V(f) = \int f d\lambda$, for any $f \in C_0(S)$, and $||L_V|| = ||\lambda||_V$. If $F \in C_0(S,X)$, then $|L(F)| \leq L_V(|F|) = \int |F| d\lambda$. Now, let $f \in C_0(S)^+$. For $x \in X$ and $\varepsilon > 0$, choose $g \in C_0(S)$, such that $|g| \leq f$ and $|\int g d\mu^{(x)}| + \varepsilon \geq \int f dV(\mu^{(x)})$. Now,

$$
\begin{aligned}
\left| \int g d\mu^{(x)} \right| &= |\ell^{(x)}(g)| \\
&= |L(gx)| \\
&\leq L_V(||x||f) \\
&= ||x||\ L_V(f) \\
&= ||x|| \int f d\lambda .
\end{aligned}
$$

Therefore, $\int f dV(\mu^{(x)}) \leq ||x|| \int f d\lambda + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\int f dV(\mu^{(x)}) \leq ||x|| \int f d\lambda$. This shows that $V(\mu^{(x)}) \leq ||x|| \lambda$. Hence for any $E \in \Sigma$, $|\mu^{(x)}(E)| \leq V(\mu^{(x)})(E) \leq ||x|| \lambda(E)$. Therefore, $||\mu_L(E)|| \leq \lambda(E)$. Since this is true for any $E \in \Sigma$, we have $V(\mu_L) \leq \lambda$. Thus μ_L is of bounded variation and is regular. Hence $\mu_L \in M(S,X^*)$, and also $||\mu_L||_V \leq ||\lambda||_V = ||L||$.

From the definitions, it is clear that $x \circ \mu_L = \mu^{(x)}$ for any $x \in X$. So if we take $F \in C_0(S,X)$ of the form $F = \sum_{i=1}^{n} f_i x_i$, with each $x_i \in X$ and each $f_i \in C_0(S)$, then from Proposition 2.5 and 2.6, it easily follows that $\int F d\mu_L = L(F)$. Since the functions of this form are dense in $C_0(S,X)$, we have $L(F) = \int F d\mu_L$ for any $F \in C_0(S,X)$. It also follows that $|L(F)| = |\int F d\mu_L| \leq |||F|||_\infty ||\mu_L||_V$, and hence $||L|| \leq ||\mu_L||_V$. Since we have already proved $||\mu_L||_V \leq ||L||$, we have $||\mu_L|| = ||L||$. Moreover, it is easy to see that the correspondence $L \to \mu_L$ is linear and our proof is complete.

Note: If G is a locally compact abelian group, then the correspondence of $C_0(G,X)^*$ and $M(G,X^*)$ will be taken in the following way for convenience,

$$\langle \mu, F \rangle = \int_G F(-x)\, d\mu(x).$$

2.9 Product Measures: Let (X,Y,Z) be a bilinear system of Banach spaces. Let S and T be locally compact Hausdorff spaces and let $S \times T$ be their cartesian product.

Let Σ_S, Σ_T and $\Sigma_{S \times T}$ denote the σ-ring of Borel sets on S, T and $S \times T$ respectively. If $\mu \in M(S,X)$ and $\nu \in M(T,Y)$, then by Theorem II.8 of [17], there exists a unique $\lambda \in M(S \times T, Z)$, such that $\lambda(A \times B) = \mu(A) . \nu(B)$, for all $A \in \Sigma_S$ and $B \in \Sigma_T$. This λ is called the product measure of μ and ν, and is denoted by $\mu \times \nu$. For $E \subset S \times T$ and $s \in S$, define $E_s = \{t \in T: (s,t) \in E\}$ and for $t \in T$, define $E^t = \{s \in S: (s,t) \in E\}$. If $E \in \Sigma_{S \times T}$, then $E_s \in \Sigma_T$ and $E^t \in \Sigma_S$ for any $s \in S$ and $t \in T$. If E, μ, ν are as above, then define $\phi_E: S \to Y$ by $\phi_E(s) = \nu(E_s)$, and $\psi_E: T \to X$ by $\psi_E(t) = \mu(E^t)$. We then have the following (Theorem III.1 of [17]).

Proposition 2.11: Let $E \in \Sigma_{S \times T}$, and μ, ν be as above. Then the following are true.

(a) If $\nu(\Sigma_T)$ is separable, then $\phi_E: S \to Y$ is μ-integrable and $\int \phi_E \, d\mu = \mu \times \nu(E)$.

(b) If $\mu(\Sigma_S)$ is separable, then $\psi_E: T \to X$ is ν-integrable and $\int \psi_E d\nu = \mu \times \nu(E)$.

2.10 Locally compact group: Convolutions: Let G be a locally compact group with a left invariant Haar measure λ. Σ will denote the σ-ring of Borel sets in G. As discussed in §2.4 and §2.7, we can form the various spaces of X-valued functions and measures on G, e.g. $L^p(G,X)$ for $1 \leq p < \infty$, $M(G,X)$, $C_o(G,X)$ and $C_c(G,X)$. As mentioned earlier, the

index 'X' will be suppressed in these notations, when $X = C$. Since λ is regular, $L^1(G,X)$ can be imbedded canonically in $M(G,X)$.

For any function F on G, and any $s \in G$, ${}_sF$ will denote the left translate of F by s, defined by ${}_sF(t) = F(st)$. Similarly, we define the right translate F_s by $F_s(t) = F(ts)$. For measures μ, the left and right translates are defined by ${}_s\mu(E) = \mu(sE)$ and $\mu_s(E) = \Delta(s^{-1})\,\mu(Es)$ for any $E \in \Sigma$. Here Δ is the modular function of G, and is introduced in the definition of μ_X, so that for $\mu \in L^1(G,X)$, the two definitions coincide. If $F \in L^p(G,X)$, then both ${}_sF$ and F_s belong to $L^p(G,X)$ for any $s \in G$. Moreover, $|||{}_sF|||_p = |||F|||_p$ and $|||F_s|||_p = [\Delta(s^{-1})]^{1/p}|||F|||_p$. It can also be proved, that the map $s \to {}_sF$ of G into $L^p(G,X)$ is right uniformly continuous, and that the map $s \to F_s$ is continuous.

If $g \in L^1(G)$ and $F \in L^p(G,X)$, then using the Bochner integral as defined in §2.3, we can define the convolution of g and F by

$$g * F(s) = \int_G g(st)F(t^{-1})d\lambda(t) = \int_G g(t)F(t^{-1}s)d\lambda(t)$$

for almost all s. $g * F$ so defined, belongs to $L^p(G,X)$, and $|||g * F|||_p \leq ||g||_1\,|||F|||_p$. Similarly, if $g \in L^1(g)$ and $F \in C_o(G,X)$, then we can define $g * F$ in the same way,

and we have $g * F \in C_o(G,X)$ with $|||g * F|||_\infty \leq ||g||_1$ $|||F|||_\infty$. In this way, $L^p(G,X)$ and $C_o(G,X)$ become left $L^1(G)$-modules.

If $F \in L^p(G,X)$ and $\Delta^{-1/p'} g \in L^1(G)$ ($p' = \frac{p}{p-1}$, $\frac{1}{p'} = 0$ if $p = 1$), then we can also define

$$F * g(s) = \int_G \Delta(t^{-1})g(t)F(st^{-1})d\lambda(t)$$

$$= \int_G g(ts)F(t^{-1}) \ \Delta(t^{-1})d\lambda(t)$$

for almost all s. $F * g \in L^p(G,X)$, and $|||F * g|||_p \leq |||F|||_p \ ||\Delta^{-1/p'}g||_1$. If support of $g \subset K_1$, and support of $F \subset K_2$, then support of $g * F \subset K_1K_2$, and support of $F * g \subset K_2K_1$. The proofs of these facts are exactly similar to the case when $X = C$ (see [13]).

2.11 <u>$L^1(G,A)$ as a Banach algebra:</u> If G is a locally compact abelian group, and A is a commutative Banach algebra, then $L^1(G,A)$ forms a commutative Banach algebra (see [10], [11] and [18]), where multiplication of two elements $F,G \in L^1(G,A)$, is defined by the convolution,

$$F * G(s) = \int_G F(st) \ G(t^{-1})d\lambda(t)$$

Let $a \in A$ and $F \in L^1(G,A)$. We define the function aF by $(aF)(s) = aF(s)$. It is easy to see that $aF \in L^1(G,A)$ and $|||aF|||_1 \leq ||a|| \ |||F|||_1$. In this way, $L^1(G,A)$ forms a Banach A-module. It is also easy to prove that for any $f \in L^1(G)$, $a \in A$ and $F \in L^1(G,A)$, $(fa) * F = a(f * F) = f * (aF)$.

Gelbaum [5] and Tomiyama [40] have shown that if A and B are commutative Banach algebras then $A \otimes_\gamma B$ forms a commutative Banach algebra whose maximal-ideal space is homeomorphic (under the w^*-topologies) to the cartesian product of the maximal-ideal spaces of A and B. The following is proved (Theorem 4 of [6]) by using the Gelfand transform.

<u>Proposition 2.12</u>. If A_1 and A_2 are commutative, semi-simple Banach algebras then $A_1 \otimes_\gamma A_2$ has an identity if and only if both A_1 and A_2 have identity.

We have already noted that as a Banach space, $L^1(G,A)$ is isometrically isomorphic to $L^1(G) \otimes_\gamma A$. If A is a commutative Banach algebra, then this isomorphism is actually a Banach algebra isomorphism. Thus the maximal-ideal space of $L^1(G,A)$ is given by $\Gamma \times M$(see also [10] and [18]), where Γ is the dual of G, and M is the maximal-ideal space of A. For $\gamma \in \Gamma$ and $m \in M$, the multiplicative linear functional (γ,m) on $L^1(G,A)$ is given by

$$(\gamma,m)(F) = m\left(\int_G \bar{\gamma}F d\lambda\right) = \int_G \bar{\gamma}(m \circ F)d\lambda$$

It is easy to prove that $||(\gamma,m)||_{L^1(G,A)^*} = ||m||_{A^*}$. Indeed, from the above it follows that $||(\gamma,m)||_{L^1(G,A)^*} \leq ||m||_{A^*}$. For the other inequality, choose $f \in L^1(G)$ such that $\hat{f}(\gamma) = \int \bar{\gamma}f d\lambda = ||f||_1 = 1$, and for arbitrary $\epsilon > 0$,

choose $a \in A$, such that, $||a|| = 1$ and $m(a) \geq ||m||_{A^*} - \varepsilon$. Then $|||fa|||_1 = 1$, and $(\gamma,m)\ (fa) = m(a) \geq ||m||_{A^*} - \varepsilon$. Since ε is arbitrary we get, $||(\gamma,m)||_{L^1(G,A)^*} \geq ||m||_{A^*}$.

The following are well-known.

Proposition 2.13: $L^1(G,A)$ is semisimple if and only if A is semisimple (see Theorem 3.1 of [18]).

Proposition 2.14: $L^1(G,A)$ is regular if and only if A is regular (see Theorem 3 of [40]).

Definition: A commutative, semisimple, regular Banach algebra is called Tauberian if the subalgebra of A, consisting of elements with compactly supported Gelfand transform, is dense in A.

Proposition 2.15: $L^1(G,A)$ is Tauberian if A is Tauberian (see the proof of the main theorem of [11]).

In Chapter III, we shall prove some other similar properties of $L^1(G,A)$.

2.12 Convolution of Measures: Let (X,Y,Z) form a Bilinear system of Banach spaces. Let $\mu \in M(G,X)$ and $\nu \in M(G,Y)$. Then, using the notion of product measures, we can define (see [17]) a convolution of μ and ν by $\mu * \nu(E) = \mu \times \nu(E_2)$, where $E \in \Sigma$, and $E_2 = \{(s,t) \in G \times G: st \in E\}$. As has been shown in Theorem IV.2 of [17], $\mu * \nu \in M(G,Z)$ and $V(\mu * \nu) \leq V(\mu) * V(\nu)$. It is also

easy to show that, $_s(\mu * \nu) = {_s\mu} * \nu$ and $(\mu * \nu)_s = \mu * \nu_s$ for any $s \in G$.

If $\nu(\Sigma)$, the range of the measure ν, is separable, then by Proposition 2.11, we get $\mu \times \nu(E_2) = \int_G \phi_{E_2}(s)d\mu(s)$. Here, $\phi_{E_2}(s) = \nu((E_2)_s)$ and $(E_2)_s = \{t \in G: (s,t) \in E_2\} = s^{-1}E$. Thus, we get,

$$\mu * \nu(E) = \int_G \nu(s^{-1}E)\ d\mu(s) \tag{2.1}$$

Similarly, if $\mu(\Sigma)$, the range of μ, is separable, then we get

$$\mu * \nu\ (E) = \int_G \mu(Et^{-1})d\nu(t) \tag{2.2}$$

If X is any Banach space, then (C,X,X) and (X,C,X) form Bilinear systems. So, given $\mu \in M(G,X)$ and $\nu \in M(G)$, we can define $\mu * \nu$ and $\nu * \mu$ as elements of $M(G,X)$. If $\mu \in M(G,X)$ and $\nu \in L^1(G)$, then by our results in Chapter IV (Theorem 4.5), it will follow that $\mu * \nu$ and $\nu * \mu$ belong to $L^1(G,X)$. We shall also see that if μ belongs to $L^1(G,X)$, then these definitions of $\mu * \nu$ and $\nu * \mu$ agree with the earlier definitions given in §2.10.

If $\mu \in M(G,X)$ and $\nu \in M(G)$, then equation (2.1) is valid, since ν is scalar-valued. Similarly, we also get

$$\nu * \mu\ (E) = \int_G \nu(E\ t^{-1})\ d\mu(t) \tag{2.3}$$

If $\mu(\Sigma)$ happens to be separable, then equation (2.2) is valid and also,

$$\nu * \mu\,(E) = \int_G \mu(s^{-1}E)\, d\nu(s) \tag{2.4}$$

However, in this case, equations (2.2) and (2.4) are valid under weaker conditions also.

Proposition 2.16: If $\mu \in M(G,X)$, and $\nu \in M(G)$, then equation (2.2) is valid whenever the function $t \to \mu(Et^{-1})$ has a separable range.

Proof: The function $t \to \mu(Et^{-1})$ is weakly ν-measurable for any measure ν. Hence, it is ν-measurable whenever it has separable range. Also,

$$\begin{aligned}\int_G ||\mu(Et^{-1})||\, dV(\nu)(t) &\leq \int_G V(\mu)(Et^{-1})\, dV(\nu)(t) \\ &= V(\mu) * V(\nu)(E).\end{aligned}$$

Hence, by Proposition 2.7, $t \to \mu(Et^{-1})$ is ν-integrable and the right hand side of equation (2.2) is defined. Now, for any $\phi \in X^*$,

$$\begin{aligned}\phi\left(\int_G \mu(Et^{-1})\, d\nu(t)\right) &= \int_G \phi \circ \mu(Et^{-1})\, d\nu(t) \\ &= (\phi \circ \mu) * \nu(E)\end{aligned}$$

From equation (2.1), it easily follows that $\phi \circ (\mu * \nu) = (\phi \circ \mu) * \nu$. Hence,

$$\phi \circ (\mu * \nu)(E) = (\phi \circ \mu) * \nu(E) = \phi\left(\int_G \mu(Et^{-1})\, d\nu(t)\right)$$

Since $\phi \in X^*$ is arbitrary, we see that equation (2.2) is valid and our proof is complete.

Similarly, we can show that equation (2.4) is valid whenever the function $s \to \mu(s^{-1}E)$ has separable range.

2.13 <u>M(G,A) as a Banach algebra:</u> Let G be a locally compact abelian group, and let A be a commutative Banach algebra. Then (A,A,A) forms a Bilinear system of Banach spaces, and given $\mu, \nu \in M(G,A)$, we can define $\mu * \nu \in M(G,A)$, which satisfies $||\mu * \nu||_v \leq ||\mu||_v \, ||\nu||_v$. We also assume that the range of every $\mu \in M(G,A)$ is separable. This is true if A has RNP, or if G is second countable. Under these conditions, the convolution so defined is associative by Theorem IV.4 of [17]. Moreover, since G is abelian, it is easy to prove that the convolution is commutative. Thus under these conditions, $M(G,A)$ is a commutative Banach algebra. The algebra $L^1(G,A)$ is a subalgebra of $M(G,A)$. Moreover from the results of Chapter V (Lemma 5.7) it will follow that $L^1(G,A)$ is an ideal in $M(G,A)$.

There is a natural Banach space isometric isomorphism from $M(G) \otimes_\gamma A$ into $M(G,A)$ (Theorem 4.2 of [22]). This is also a Banach algebra isomorphism. If A has RNP, then this isomorphism is onto (Theorem 4.4 of [22]).

2.14 <u>Module Tensor Products and Multipliers:</u> Let A be a Banach algebra. A Banach space V is said to be a left Banach A-module, if there exists a mapping $(a,v) \to a \circ v$ from

$A \times V$ into V, such that V is an algebraic left module over A with respect to o, and for which $||aov|| \leq ||a|| \, ||v||$, for all $a \in A$ and $v \in V$. Similarly, we can define right Banach A-modules. For a locally compact group G, the spaces $M(G,X)$, $L^p(G,X)$, $C_0(G,X)$ are left Banach $L^1(G)$-modules under convolution.

If V and W are A-modules, then the A-module tensor product $V \otimes_A W$ is defined (see [31] and [32]) to be the quotient Banach space $V \otimes_\gamma W/K$, where K is the closed linear subspace of $V \otimes_\gamma W$, spanned by the elements of the form $(aov) \otimes \omega - v \otimes (aow)$, with $a \in A$, $v \in V$ and $\omega \in W$.

A continuous linear transformation from V to W is called an A-module homomorphism, if $T(aov) = aoT(v)$, for all $a \in A$ and $v \in V$. The space of A-module homomorphisms from V to W, denoted by $\mathrm{Hom}_A(V,W)$, is a Banach space under the operator norm. $\mathrm{Hom}_A(V,V)$ will be denoted by $\mathrm{Hom}_A(V)$. Moreover, we will write $\otimes_G$ for $\otimes_{L^1(G)}$, and Hom_G for $\mathrm{Hom}_{L^1(G)}$.

Rieffel has shown (2.12 and 2.13 of [31]), that there is a natural isometric isomorphism between $\mathrm{Hom}_A(V,W^*)$ and $(V \otimes_A W)^*$, under which, the linear functional on $(V \otimes_A W)$, which corresponds to an operator $T \in \mathrm{Hom}_A(V,W^*)$, has the value $\langle T(v), \omega \rangle$, at the element $v \otimes \omega$ of $V \otimes_A W$. We shall make a crucial use of this in Chapter V.

Let A be a commutative, semisimple Banach algebra. A bounded linear operator T of A is called a multiplier if for any $f, g \in A$, $T(fg) = fT(g)$ (see [23]). Now A forms a left or right Banach A-module under multiplication. Obviously the multipliers of A are nothing but the A-module homomorphisms of A. If T is a multiplier of A, then there exists (see [23]) a continuous function $\hat{T}$ (called the Gelfand Transform of T) on $\mathcal{M}$, the maximal ideal space of A, such that $\widehat{T(a)}\,(m) = \hat{a}(m)\,\hat{T}(m)$ for all $a \in A$ and $m \in \mathcal{M}$, where $\hat{a}$ denotes the Gelfand transform of a. Also $|\hat{T}(m)| \leq ||T||$ for any $m \in \mathcal{M}$. In Chapter V we shall determine the multipliers of $L^1(G,A)$, and in Chapter VI we shall prove some results on multipliers.

CHAPTER III

$L^1(G,A)$ AS A GROUP ALGEBRA

3.1 Throughout this chapter, G will denote a locally compact abelian group with dual Γ, and A will be a commutative, semisimple Banach algebra with the maximal-ideal space M. It is well-known (Theorem 3.2 of [18]) that if $A = L^1(H)$, for some locally compact abelian group H, then $L^1(G,A)$ is isometrically isomorphic to $L^1(G \times H)$. The main result of this chapter is a converse of this. We first prove certain simple properties of $L^1(G,A)$. We include these preliminary results only for the sake of completeness. Though we did not see these in the literature, these may be well-known otherwise.

Theorem 3.1 $L^1(G,A)$ has a bounded approximate identity if and only if A has a bounded approximate identity.

Proof: Let A have a bounded approximate identity with bound M. Let $F \in L^1(G,A)$ and $\varepsilon > 0$. Choose $a_i \in A$ and $f_i \in L^1(G)$ for $1 \leq i \leq n$, such that $\left\|F - \sum_{i=1}^{n} f_i a_i\right\|_1 \leq \varepsilon$. Let $M_1 = \max_{1 \leq i \leq n} \|a_i\|$, and $M_2 = \max_{1 \leq i \leq n} \|f_i\|_1$. Choose $b \in A$ with $\|b\| \leq M$, such that $\|ba_i - a_i\| \leq \frac{\varepsilon}{nM_2}$, for $1 \leq i \leq n$. Choose $f \in L^1(G)$ with $\|f\|_1 = 1$, such that $\|f * f_i - f_i\|_1 \leq \frac{\varepsilon}{nM_1}$, for $1 \leq i \leq n$. Take $F_1 = fb$. Then

$F_1 \in L^1(G,A)$ and $|||F_1|||_1 \leq M$. Moreover, an easy computation shows that $|||F_1 * F - F|||_1 \leq (M+3)\,\varepsilon$. Since ε is arbitrary, we see that $L^1(G,A)$ has a bounded approximate identity with bound M.

Conversely, let $L^1(G,A)$ have a bounded approximate identity with bound M. Let $a \in A$ and $\varepsilon > 0$. Choose any $f \in L^1(G)$, such that f is nonnegative with $||f||_1 = \int_G f d\lambda = 1$. Since $L^1(G,A)$ has a bounded approximate identity, there exists $F \in L^1(G,A)$ with $|||F|||_1 \leq M$, such that $|||(fa) * F - (fa)|||_1 \leq \varepsilon$. Let $b = \int_G F d\lambda$. Then $b \in A$ with $||b|| \leq M$. Moreover, by Theorem 2.1 of [18],

$$\begin{aligned}\int_G (fa) * F \, d\lambda &= \left(\int_G fa d\lambda\right)\left(\int_G F d\lambda\right)\\ &= ab\end{aligned}$$

Hence,

$$\begin{aligned}||ab-a|| &= \left|\left|\int_G (fa) * F d\lambda - \int_G fa d\lambda\right|\right|\\ &\leq \int_G ||(fa) * F - (fa)||\, d\lambda\\ &= |||(fa) * F - (fa)|||_1 \leq \varepsilon .\end{aligned}$$

This proves that A has a bounded approximate identity with bound M and the proof of the theorem is complete.

The following is the converse of Proposition 2.15.

<u>Theorem 3.2</u> If $L^1(G,A)$ is Tauberian then A is Tauberian.

<u>Proof:</u> Let $a \in A$ and $\varepsilon > 0$. Take $f \in L^1(G)$, such that f is nonnegative with $||f||_1 = \int_G f d\lambda = 1$. Since $L^1(G,A)$

is Tauberian, there exists $F \in L^1(G,A)$, such that the Gelfand transform of F has compact support K, and $|||(fa) - F|||_1 < \varepsilon$. Take $K_1 = \{m \in M : (1,m) \in K\}$. We note that $K \subset \Gamma \times M$, the maximal-ideal space of $L^1(G,A)$, and 1 denotes the identity element of Γ. Obviously K_1 is compact, and if we take $b = \int_G F d\lambda$, then it is easy to see that the Gelfand transform of b, is supported in K_1. Moreover,

$$\begin{aligned} ||a-b|| &= ||a\int_G f d\lambda - \int_G F d\lambda|| \\ &\leq \int_G ||(fa) - F|| d\lambda \\ &= |||(fa) - F|||_1 \leq \varepsilon . \end{aligned}$$

Hence A is Tauberian, and this completes the proof.

3.2 <u>$L^1(G,A)$ as a group algebra:</u> We shall now prove,

<u>Theorem 3.3:</u> $L^1(G,A)$ is isometrically isomorphic to $L^1(H)$ for some locally compact abelian group H, if and only if A is isometrically isomorphic to $L^1(G_1)$ for some locally compact abelian group G_1.

To prove this theorem we shall use the result of [30], where a characterisation of the group algebra of a locally compact abelian group, is given. We need some definitions (see [30]). Hereafter we shall write 'm.l.f.' to mean a multiplicative linear functional.

Let m be any non-zero m.l.f. of A. Define $P_m = \{a \in A : m(a) = ||m|| \, ||a||\}$. It is easy to prove that

P_m is a norm-closed cone of A, such that $P_m \cap (-P_m) = \{0\}$. By the order induced by m, we shall mean the partial order induced by P_m on A. We define $R_m = \{a-b: a,b \in P_m\}$. A m.l.f. m of A is said to be L'-inducing, if the following conditions are satisfied.

(1) $||m|| = 1$

(2) P_m is a lattice under the order induced by m. This will imply that R_m is also a lattice (see page 34 of [30]).

(3) If $a,b \in R_m$ and $a \wedge b = 0$, then $||a+b|| = ||a-b||$

(4) For any $a \in A$, there exist unique a_1, $a_2 \in R_m$, such that $a = a_1 + ia_2$. (We shall write $a_1 = \mathrm{Re}(a)$ and $a_2 = \mathrm{Im}(a)$.)

(5) Defining $|a| = \bigvee\{\mathrm{Re}\,(e^{i\theta} a): \theta \in [0, 2\pi]\}$, we have $||a|| = ||\,|a|\,||$.

We note that if (1) - (3) are satisfied, then R_m forms a real abstract L-space in the sense of Kakutani [19], and hence R_m is boundedly lattice complete (see page 35 of [30]). Therefore $|a|$ is well-defined.

In [30], a L'-inducing m.l.f. is defined to be a m.l.f. which satisfies the following condition in addition to (1) - (5).

(6) For any $a,b \in A$, $|a * b| \leq |a| * |b|$, where $*$ denotes the product of A.

However, White [43] has shown that a m.l.f. satisfying (1) - (5) automatically satisfies (6), and hence our definition is equivalent to that of [30].

We can now state Rieffel's characterisation of the group algebra of a locally compact abelian group.

<u>Theorem R1:</u> Let A be a commutative, semisimple Banach algebra. A is isometrically isomorphic to $L^1(H)$ for a locally compact abelian group H, if and only if,

(a) every m.l.f. of A is L'-inducing

(b) A is Tauberian.

We are now in a position to prove Theorem 3.3.

<u>Proof of Theorem 3.3:</u> The 'if' part follows from Theorem 3.2 of [18]. Here, we shall prove the 'only if' part. Let $L^1(G,A)$ be isometrically isomorphic to $L^1(H)$, for a locally compact abelian group H. Then $L^1(G,A)$ is Tauberian. Hence, by Theorem 3.2, A is Tauberian. Thus, in view of Theorem R1, our proof will be complete, if we can prove that every m.l.f. of A is L'-inducing.

Let m be any m.l.f. of A. Then, we have a m.l.f. (1,m) of $L^1(G, A)$ (see the proof of Theorem 3.2), given by

$$(1,m)(F) = m(\int_G F d\lambda) = \int_G (m \circ F) d\lambda$$

By our hypothesis and Theorem R1, (1,m) is L'-inducing. Hence, $||(1,m)||_{L^1(G,A)^*} = 1$. But $||m||_{A^*} = ||(1,m)||_{L^1(G,A)^*}$ (see §2.11). Hence $||m||_{A^*} = 1$, i.e. m satisfies (1).

Let us now choose some fixed $f \in L^1(G)$, such that f is nonnegative with $||f||_1 = \int_G f d\lambda = 1$. Let $a \in P_m$. Then $fa \in P_{(1,m)}$, because

$$\begin{aligned}(1,m)(fa) &= m \int_G (fa) d\lambda \\ &= m(a) \int_G f d\lambda \\ &= ||a|| \\ &= |||fa|||_1\end{aligned}$$

This implies that if $a \geq b$ then $fa \geq fb$, where the orders are those induced by m and $(1,m)$ respectively. On the otherhand, if $F \in P_{(1,m)}$ then $\int_G F d\lambda \in P_m$, because

$$\begin{aligned}\Big|\Big|\int_G F d\lambda\Big|\Big| &\geq m(\int_G F d\lambda) \\ &= (1,m)(F) \\ &= |||F|||_1 \\ &\geq \Big|\Big|\int_G F d\lambda\Big|\Big|\end{aligned}$$

This implies that if $F \geq F_1$ then $\int_G F d\lambda \geq \int_G F_1 d\lambda$. Now, let $a, b \in P_m$. Then $(fa), (fb) \in P_{(1,m)}$. Since $P_{(1,m)}$ is a lattice, the least upper bound $(fa) \vee (fb)$ exists. Let $c = \int_G (fa) \vee (fb) d\lambda$. Then,

$$\begin{aligned}c - a &= \int_G (fa) \vee (fb) d\lambda - \int_G (fa) d\lambda \\ &= \int_G [(fa) \vee (fb) - (fa)] d\lambda\end{aligned}$$

Since $(fa) \vee (fb) - (fa) \in P_{(1,m)}$, we have $c - a \in P_m$, and hence

$c \geq a$. Similarly, we can prove that $c \geq b$. Also if $d \geq a$ and $d \geq b$, then $fd \geq fa$ and $fd \geq fb$. Therefore $fd \geq (fa) \vee (fb)$. Therefore, $d = \int_G (fd) d\lambda \geq \int_G (fa) \vee (fb) d\lambda = c$. Thus, the least upper bound $a \vee b$ exists, and we have

$$a \vee b = \int_G (fa) \vee (fb) d\lambda$$

Note, that the least upper bound $(a \vee b)$, if it exists, depends only on a and b. Hence $\int_G (fa) \vee (fb) d\lambda$ is independent of the choice of f. Similarly, we can prove that $a \wedge b$ exists, and $a \wedge b = \int_G (fa) \wedge (fb) d\lambda$. Thus, P_m is a lattice with respect to the order induced by m. Hence m satisfies (2).

Let $a, b \in P_m$. Since $a \vee b \geq a$, we have $f(a \vee b) \geq fa$. Similarly, $f(a \vee b) \geq fb$. Therefore, $f(a \vee b) \geq (fa) \vee (fb)$. Hence,

$$\begin{aligned} |||f(a \vee b) - (fa) \vee (fb)|||_1 &= (1,m) \left[f(a \vee b) - (fa) \vee (fb) \right] \\ &= m \left\{ \int_G \left[f(a \vee b) - (fa) \vee (fb) \right] d\lambda \right\} \\ &= m \left[(a \vee b) - \int_G (fa) \vee (fb) d\lambda \right] \\ &= 0. \end{aligned}$$

Therefore, $f(a \vee b) = (fa) \vee (fb)$. Similarly, $f(a \wedge b) = (fa) \wedge (fb)$. It is easy to show that if $F \in R_{(1,m)}$ then $\int_G F d\lambda \in R_m$, and if $a \in R_m$ then $fa \in R_{(1,m)}$. Moreover, all the above relations for $a \vee b$ and $a \wedge b$ are true for $a, b \in R_m$.

Now, let $a, b \in R_m$ such that $a \wedge b = 0$. Then fa, $fb \in R_{(1,m)}$ and $(fa) \wedge (fb) = f(a \wedge b) = 0$. Since $(1,m)$ is L'-inducing, we have $|||fa + fb|||_1 = |||fa - fb|||_1$. Therefore $||a+b|| = ||a-b||$, and hence m satisfies (3).

Next, let $a \in A$. Consider $fa \in L^1(G,A)$. Since $(1,m)$ is L'-inducing, we can write $fa = F_1 + iF_2$ with $F_1, F_2 \in R_{(1,m)}$. Then $\int_G F_i d\lambda \in R_m$ for $i = 1,2$, and $a = \int_G (fa) d\lambda = \int_G F_1 d\lambda + i \int_G F_2 d\lambda$. Also, let $a = a_1 + ia_2 = a_3 + ia_4$ with $a_i \in R_m$ for $i = 1,2,3,4$. Then, each $fa_i \in R_{(1,m)}$, and $fa = fa_1 + ifa_2 = fa_3 + ifa_4$. Since $(1,m)$ is L'-inducing, $fa_1 = fa_3$ and $fa_2 = fa_4$. Hence, $a_1 = \int_G (fa_1) d\lambda = \int_G (fa_3) d\lambda = a_3$, and similarly $a_2 = a_4$. This proves that m satisfies (4). We have also proved that $Re(a) = \int_G Re\ (fa) d\lambda$, and $Re(fa) = f\ Re(a)$.

To prove that m satisfies (5), we shall first prove $|a| = \int_G |fa| d\lambda$ and $|fa| = f|a|$. We have,

$$\begin{aligned} \int_G |fa| d\lambda - Re(e^{i\theta} a) &= \int_G |fa| d\lambda - \int_G Re(e^{i\theta} fa) d\lambda \\ &= \int_G [\,|fa| - Re(e^{i\theta} fa)\,] d\lambda \\ &\geq 0 \qquad \text{since } |fa| \geq Re(e^{i\theta} fa). \end{aligned}$$

Therefore, $\int_G |fa| d\lambda \geq Re(e^{i\theta} a)$ for any $\theta \in [0, 2\pi]$. Hence $\int_G |fa| d\lambda \geq |a|$. On the otherhand, $|a| \geq Re\ (e^{i\theta} a)$. Therefore, $f|a| \geq f\ Re(e^{i\theta} a) = Re(e^{i\theta} fa)$. Since this is true for any $\theta \in [0, 2\pi]$, we have $f|a| \geq |fa|$. Hence $|a| = \int_G f|a| d\lambda \geq \int_G |fa| d\lambda$. Therefore, we have proved $|a| = \int_G |fa| d\lambda$. To

prove the second relation, we note that we have already seen $f|a| \geq |fa|$. Therefore,

$$|||f|a| - |fa|\ |||_1 = (1,m)\ (f|a| - |fa|\ )$$
$$= m\ [\int_G (f|a| - |fa|)d\lambda\ ]$$
$$= m\ (|a| - \int_G |fa|\, d\lambda)$$
$$= 0$$

Hence, $|fa| = f|a|$.

Now, since $(1,m)$ is L'-inducing, we have $|||fa|||_1 = |||\ |fa|\ |||_1$. Therefore,

$$||a|| = |||fa|||_1$$
$$= |||\ |fa|\ |||_1$$
$$= |||\ |a|f\ |||_1$$
$$= ||\ |a|\ ||$$

Hence m satisfies (5). Thus we have proved that m satisfies (1) - (5), and hence m is $\bar{L}'$-inducing. Since m is an arbitrary m.l.f. of A, the proof of the theorem is complete.

3.3 <u>M(G,A) as a measure algebra</u>: Now, we shall try to investigate conditions, under which $M(G,A)$ is isometrically isomorphic to $M(H)$, for some locally compact abelian group H. First, we prove the following theorem.

Theorem 3.4. If $A = M(H)$, for some locally compact abelian group H, then $M(G,A)$ is isometrically isomorphic to $M(G \times H)$.

Proof: Since $M(H)$ is the dual of $C_0(H)$, by Theorem 2.1 $M(G,M(H))$ is nothing but the dual of $C_0(G,C_0(H))$. On the otherhand, $M(G \times H)$ is the dual of $C_0(G \times H)$. Hence the theorem will be proved, if we prove that $C_0(G,C_0(H))$ is isometrically isomorphic to $C_0(G \times H)$. Let $f \in C_0(G \times H)$. For $s \in G$, consider the function f_s on H defined by $f_s(h) = f(s,h)$. Obviously $f_s \in C_0(H)$. Define the $C_0(H)$-valued function F on G by $F(s) = f_s$. Then it is easy to see that $F \in C_0(G,C_0(H))$, and that the association of F with f gives an isometric isomorphism between $C_0(G \times H)$ and $C_0(G,C_0(H))$. This completes the proof.

In Theorem B of [30], Rieffel has given sufficient conditions for a commutative Banach algebra A to be isometrically isomorphic to $M(H)$, for a locally compact abelian group H. We shall use this to prove,

Theorem 3.5: If $M(G) \otimes_\gamma A$ is isometrically isomorphic to $M(G_1)$ for some locally compact abelian group G_1, then A is isometrically isomorphic to $M(H)$ for some locally compact abelian group H.

Under certain conditions (see §2.13) $M(G) \otimes_\gamma A$ is isometrically isomorphic to $M(G,A)$. In any case, $M(G) \otimes_\gamma A$ can always be imbedded in $M(G,A)$. Hence, Theorem 3.5 can be

taken to be a partial converse of Theorem 3.4. Before proving Theorem 3.5, we state Rieffel's result which we need.

Let D be the collection of L'-inducing m.l.f.'s of A. Consider the w^*-topology on D. A continuous function p on D, is said to be a D-Eberlein function, if there exists a constant $k > 0$, such that, for any choice of points $m_1, m_2, \ldots, m_n$ of D, and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\left| \sum_{i=1}^{n} \alpha_i \, p(m_i) \right| \leq k \left\| \sum_{i=1}^{n} \alpha_i \, m_i \right\|_{A^*}.$$

The following theorem gives necessary and sufficient conditions for a commutative Banach algebra to be isometrically isomorphic to the measure algebra of a locally compact abelian group.

<u>Theorem R2</u>: Let A be a commutative Banach algebra, and let D be the set of L'-inducing m.l.f.'s of A. Then, A is isometrically isomorphic to M(H) for some locally compact abelian group H, if and only if

(1) D is a separating family of linear functionals of A.

(2) D is locally compact in the w^*-topology.

(3) every D-Eberlein function is the restriction to D of the Gelfand transform of some element of A.

The 'if' part is nothing but Theorem B of [30]. The 'only if' part follows from the following and the familiar properties of Fourier-Stieltjes transforms.

Theorem 3.6: The L'-inducing m.l.f.'s of M(G) are precisely those given by Γ, the dual of G.

Proof: Let S be the structure semigroup of M(G) (see §4.3 of [39]). M(G) can be identified (§3.2 of [39]) with a weak*-dense subalgebra of M(S) and under this identification, the m.l.f.'s of M(G) are given by $\hat{S}$, the collection of semicharacters of S. Let $f \in \hat{S}$. Then, using the same arguments used in Proposition 2.5 of [30] (see also Proposition 2.8 of [30]), we can prove that f represents a L'-inducing m.l.f. if and only if $|f(s)| = 1$ for all $s \in S$. By 4.3.3 of [39], $\{f \in \hat{S}: |f| = 1\}$ is the canonical image of Γ in $\hat{S}$, and hence our theorem is proved.

Throughout the rest of this chapter we shall identify $M(G) \otimes_{\gamma} A$ with its canonical image in $M(G,A)$ (see §2.13). Under this identification, an element of $M(G) \otimes_{\gamma} A$, of the form $\sum_{i=1}^{\infty} \mu_i \otimes a_i$ with $\mu_i \in M(G)$ and $a_i \in A$, is identified with the A-valued measure $\sigma = \sum_{i=1}^{\infty} \mu_i a_i$. Taking the positive measure $\nu = \sum_{i=1}^{\infty} ||a_i|| \; v(\mu_i)$, it is easy to see that σ has a derivative with respect to ν. Thus, every element of $M(G) \otimes_{\gamma} A$ has a derivative with respect to some positive measure in M(G). If $\gamma \in \Gamma$ and $m \in M$, then under the above identification, the action of the m.l.f. (γ,m) on $M(G) \otimes_{\gamma} A$, is given by,

$$(\gamma,m)(\sigma) = m\left(\int_G \bar{\gamma}\, d\sigma\right) = \int_G \bar{\gamma}\, d(m \circ \sigma).$$

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5: Let M_G denote the maximal-ideal space of $M(G)$. We have $\Gamma \subset M_G$ and, by Theorem 3.6, Γ is the collection of L'-inducing m.l.f.'s of $M(G)$. The maximal-ideal space of $M(G) \otimes_\gamma A$ is then given by $M_G \times M$, with the w*-topology coinciding with the product of the w*-topologies of M_G and M. The collection of L'-inducing m.l.f.'s of A will be denoted by D, and that of $M(G) \otimes_\gamma A$ will be denoted by $D_\otimes$.

By our hypothesis, $M(G) \otimes_\gamma A$ has an identity. Hence, by Proposition 2.12, A has an identity which will be denoted by e. The identity of $M(G)$ is denoted by δ. The identity of $M(G) \otimes_\gamma A$ is then the measure δe which corresponds to $\delta \otimes e$.

We shall first prove $D_\otimes = \Gamma \times D$. For this, it is sufficient to prove the following.

(I) Let $f \in M_G$ and $m \in M$, such that $(f,m) \in D_\otimes$. Then f and m are L'-inducing, i.e. $f \in \Gamma$ and $m \in D$.

(II) Let $m \in D$ and $\gamma \in \Gamma$. Then $(\gamma,m) \in D_\otimes$.

The proofs of (I) and (II) are similar to the proof of Theorem 3.3., where we proved that m is a L'-inducing m.l.f. of A, using the fact that $(1,m)$ is a L'-inducing m.l.f. of $L^1(G,A)$. We note that all the m.l.f.'s of the algebras involved here, have norm one, since all these algebras have identity. Now to prove the first part of (I), we shall have to first

prove the following: (i) If $\mu \in P_f$ then $\mu e \in P_{(f,m)}$, and (ii) if $\sigma \in P_{(f,m)}$ then $m_o \sigma \in P_f$. Since (f,m) is L^1-inducing, $P_{(f,m)}$ is a lattice under the order induced by (f,m). Using this, we will be able to prove that P_f is a lattice under the order induced by f, with $\mu_1 \vee \mu_2 = m_o[(\mu_1 e) \vee (\mu_2 e)]$ for $\mu_1, \mu_2 \in P_f$. In this way, following the lines of the proof of Theorem 3.3, we will be able to prove the first part of (I). The proof of the second part of (I) is exactly similar. In the case of (II), we shall have to use the fact that every $\sigma \in M(G) \otimes_\gamma A$ has a derivative with respect to some positive measure. Thus, if $d\sigma = F d\nu$ for some positive measure ν and some $F \in L^1(G,\Sigma,\nu,A)$, then we shall have $\sigma \in P_{(\gamma,m)}$ if and only if $(\bar{\gamma}F)(s) \in P_m$ a.e.(ν). If $F_1, F_2 \in L^1(G,\Sigma,\nu,A)$, with $F_i(s) \in P_m$ a.e. (ν) for $i = 1,2$; then using the continuity and other properties of the lattice operations, it is easy to prove that the function $F_1 \vee F_2$ defined a.e.(ν) by $(F_1 \vee F_2)(s) = F_1(s) \vee F_2(s)$, belongs to $L^1(G,\Sigma,\nu,A)$. Using this, we shall be able to prove that if $\sigma_1, \sigma_2 \in P_{(\gamma,m)}$, with $d\sigma_i = F_i d\nu$ for $i = 1,2$, then $\sigma_1 \vee \sigma_2$ exists, and is given by

$$d(\sigma_1 \vee \sigma_2) = \gamma[(\bar{\gamma}F_1) \vee (\bar{\gamma}F_2)]d\nu .$$

We omit the rest of the details of the proofs of (I) and (II), as these are nothing but repetitions of the arguments used in the proof of theorem 3.3.

Thus we have $D_{\otimes} = \Gamma \times D$. We shall now prove that D satisfies conditions (1) - (3) of Theorem R2. Let $a_1, a_2 \in A$, such that $a_1 \neq a_2$. Then $\delta a_1 \neq \delta a_2$. By our hypothesis and Theorem R2, $D_{\otimes}$ satisfies (1)-(3) of Theorem R2. Hence by condition (1), there exists $(\gamma, m) \in D_{\otimes}$, such that $(\gamma, m)(\delta a_1) \neq (\gamma, m)(\delta a_2)$ i.e. $m(a_1) \neq m(a_2)$. Since $D_{\otimes} = \Gamma \times D$, $m \in D$, and hence D satisfies condition (1) of Theorem R2. The w^*-topology of $D_{\otimes} = \Gamma \times D$ coincides with the product of the w^*-topologies of Γ and D. Since $D_{\otimes}$ satisfies condition (2) of Theorem R2, $\Gamma \times D$ is locally compact. Hence D is locally compact in the w^*-topology, i.e. D satisfies condition (2) of Theorem R2. Finally, let p be a D-Eberlein function. Define the function P on $D_{\otimes} = \Gamma \times D$, by $P(\gamma, m) = p(m)$. Obviously, P is continuous. Moreover,

$$\left| \sum_{i=1}^{n} \alpha_i P(\gamma_i, m_i) \right| = \left| \sum_{i=1}^{n} \alpha_i p(m_i) \right|$$
$$\leq k \left\| \sum_{i=1}^{n} \alpha_i m_i \right\|_{A^*}$$

However, for any $a \in A$

$$\left\langle \sum_{i=1}^{n} \alpha_i m_i, a \right\rangle = \left\langle \sum_{i=1}^{n} \alpha_i (\gamma_i, m_i), \delta a \right\rangle$$
$$\leq \left\| \sum_{i=1}^{n} \alpha_i (\gamma_i, m_i) \right\|_{(M(G) \otimes_{\gamma} A)^*} \|\delta a\|$$
$$= \left\| \sum_{i=1}^{n} \alpha_i (\gamma_i, m_i) \right\|_{(M(G) \otimes_{\gamma} A)^*} \|a\|$$

Therefore,

$$\Big|\sum_{i=1}^{n} \alpha_i P(\gamma_i, m_i)\Big| \leq k \,\Big|\Big|\sum_{i=1}^{n} \alpha_i m_i\Big|\Big|_{A^*}$$
$$\leq k \,\Big|\Big|\sum_{i=1}^{n} \alpha_i (\gamma_i, m_i)\Big|\Big|_{(M(G)\, \otimes_\gamma A)^*}$$

Hence, P is a $D_\otimes$ -Elerbin function. But $D_\otimes$ satisfies condition (3) of Theorem R2. Therefore, there exists $\sigma \in M(G) \otimes_\gamma A$, such that, $(\gamma,m)(\sigma) = P(\gamma,m)$, for any $\gamma \in \Gamma$ and $m \in D$. Hence, taking $a = \int_G d\sigma$, we have $a \in A$, and for any $m \in D$,

$$\begin{aligned} m(a) &= m(\int_G d\sigma) \\ &= (1,m)(\sigma) \\ &= P(1,m) \\ &= p(m). \end{aligned}$$

This proves that D satisfies the condition (3) of Theorem R2. Thus D satisfies all the conditions of Theorem R2, and hence A is isometrically isomorphic to $M(H)$ for some locally compact abelian group H. This completes the proof of Theorem 3.5.

A Remark. While the thesis was being typed, Dr. U.B. Tewari pointed out that one part of Theorem 3.1 follows from a more general result on tensor products, proved by J. Holub in the paper, 'Bounded Approximate identities and tensor products', Bull. Austral. Math. Soc. 7(1972) 443-445. Earlier, we were not aware of this result.

CHAPTER IV

CONTINUOUSLY TRANSLATING MEASURES

4.1 In this chapter, we prove that the continuously translating elements of $M(G,X)$, are precisely the elements of $L^1(G,X)$. To prove this, we shall first characterise the relatively compact subsets of $L^p(G,X)$. Throughout this chapter, G will be a locally compact group with identity e, and X will be an arbitrary Banach space. We shall use the same symbol for any $F \in L^1(G,X)$, and its canonical image in $M(G,X)$. Thus, in our notation $F(E) = \int_E F(s)\, d\lambda(s)$, for any $E \in \Sigma$. Moreover, the sets contained in Σ will also be referred to as measurable sets.

4.2 Relatively compact subsets of $L^p(G,X)$. We now prove a theorem which characterises the relatively compact subsets of $L^p(G,X)$ for $1 \leq p < \infty$.

Theorem 4.1. A subset $\mathcal{F}$ of $L^p(G,X)$ is relatively compact if and only if the following conditions are satisfied.

(1) $\mathcal{F}$ is norm bounded, i.e. there exists a constant $M > 0$, such that for any $F \in \mathcal{F}$, $|||F|||_p \leq M$.

(2) Given $\varepsilon > 0$, there exists a compact set $K \subset G$, such that $\sup\{\int_{G-K} |F|^p d\lambda : F \in \mathcal{F}\} < \varepsilon$.

(3) Given $\varepsilon > 0$, there exists a neighbourhood U of identity e in G, such that $\sup \{ |||{}_aF-F|||_p : a \in U, F \in \mathcal{F} \} < \varepsilon$.

(4) For each measurable, relatively compact subset E of G, the set $\{ \int_E F(s)d\lambda(s) : F \in \mathcal{F} \}$ is relatively compact in X.

[Note that $\int_E F(s)d\lambda(s)$ is defined even for $p > 1$, since by Holder's inequality, $\int_E |F|(s)\, d\lambda(s) \leq |||F|||_p [\lambda(E)]^{1/p'}$.]

Proof. The necessity of (1) - (3) follows easily from total boundedness of $\mathcal{F}$. For (4), it is enough to note that the mapping $F \to \int_E F(s)\, d\lambda(s)$ is continuous from $L^p(G,X)$ into X.

For sufficiency, we shall construct a 5ε-net in $\mathcal{F}$ for any $\varepsilon > 0$. Choose a compact set K for ε^p as in (2), and a compact symmetric neighbourhood U for ε as in (3). Choose a continuous non-negative function g on G, supported in U, with $\int_G g\, d\lambda = 1$. For $F \in \mathcal{F}$, let $F^* = \chi_K F$ and $F^{**} = g * F^*$. Then $|||F-F^*|||_p = [\int_{G-K} |F|^p\, d\lambda]^{1/p} \leq \varepsilon$. Also

$$\begin{aligned} ||g * F(s)-F(s)|| &= || \int_G g(t)F(t^{-1}s)d\lambda(t) - \int_G g(t)F(s)d\lambda(t)|| \\ &\leq \int_G ||F(t^{-1}s)-F(s)|| g(t)d\lambda(t) \\ &\leq [\int_G ||({}_{t^{-1}}F-F)(s)||^p g(t)d\lambda(t)]^{1/p}. \end{aligned}$$

Note that $\int_G g d\lambda = 1$.

Thus,

$$|||g * F-F|||_p \leq \left[\int_G d\lambda(s) \int_G ||({}_{t^{-1}}F-F)(s)||^p g(t)d\lambda(t) \right]^{1/p}$$

$$= \left[\int_U g(t)d\lambda(t) \int_G ||{}_{t^{-1}}F-F||^p \; d\lambda \right]^{1/p}$$

$$\leq \left[\varepsilon^p \int_U g(t)d\lambda(t) \right]^{1/p}$$

$$= \varepsilon .$$

Therefore,

$$|||F^{**}-F|||_p \leq |||g * F^{*}-g * F|||_p + |||g * F-F|||_p$$

$$\leq ||g||_1 \; |||F^{*}-F|||_p + \varepsilon .$$

$$\leq 2\varepsilon .$$

Let $\mathcal{F}^{**}$ denote the family of functions F^{**} for $F \in \mathcal{F}$. In view of the above inequality, an ε-net in $\mathcal{F}^{**}$ will give a 5ε-net in $\mathcal{F}$.

To obtain an ε-net in $\mathcal{F}^{**}$, we first prove that $\mathcal{F}^{**}$ is an equicontinuous family of functions. Suppose $\varepsilon_1 > 0$. Let $M_o = \sup_{s \in K} \left[\Delta(s^{-1}) \right]$. Choose a neighbourhood V of e in G, such that $||{}_a g-g||_{p'} \leq \dfrac{\varepsilon_1}{MM_o^{1/p}}$ for all $a \in V$. Then for any $F^{**} \in \mathcal{F}^{**}$, $a \in V$ and $s \in G$, we have

$$||F^{**}(as)-F^{**}(s)|| = || \int_G \left[g(ast)-g(st) \right] F^{*}(t^{-1})d\lambda(t)||$$

$$\leq \int_G |({}_{as}g-{}_{s}g)(t)| \; ||F^{*}(t^{-1})||d\lambda(t)$$

$$\leq ||{}_{as}g-{}_{s}g||_{p'} \left[\int_G ||F^{*}(t^{-1})||^p d\lambda(t) \right]^{1/p}$$

$$= \| {}_s({}_a g - g) \|_{p'} \left[\int_K \| F^*(t) \|^p \Delta(t^{-1}) d\lambda(t) \right]^{1/p}$$

$$\leq \| {}_a g - g \|_{p'} \, M_o^{1/p} \left[\int_K \| F^*(t) \|^p d\lambda(t) \right]^{1/p}$$

$$\leq \frac{\varepsilon_1}{M M_o^{1/p}} M_o^{1/p} \; |||F^*|||_p$$

$$\leq \varepsilon_1 .$$

This proves equicontinuity of $\mathcal{F}^{**}$. Now, we shall prove that for any $s \in G$, the set $\{ F^{**}(s) : F \in \mathcal{F} \}$ is relatively compact in X. We shall construct a $3\varepsilon_2$-net in this set for any $\varepsilon_2 > 0$. Consider the function g which is positive and continuous on G, and supported in U. Let $M' = \sup_{s \in U} [\Delta(s^{-1})]$. Let $h' = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where E_i's are disjoint, measurable, relatively compact subsets of U, such that

$$\| h' - g\Delta \|_{p'} \leq \frac{\varepsilon_2}{M M_o^{1/p} M'} .$$

Let $h = h' \Delta^{-1}$. Then

$$\| h - g \|_{p'} = \| (h' - g\Delta)\Delta^{-1} \|_{p'} \leq \frac{\varepsilon_2}{M M_o^{1/p}} .$$

Now for any $s \in G$, and $F \in \mathcal{F}$,

$$\| g * F^*(s) - h * F^*(s) \| \leq \int_G |(g-h)(st)| \; \| F^*(t^{-1}) \| \; d\lambda(t)$$

$$\leq \| {}_s(g-h) \|_{p'} \left[\int_G \| F^*(t^{-1}) \|^p d\lambda(t) \right]^{1/p}$$

$$= \| (g-h) \|_{p'} \left[\int_K \| F^*(t) \|^p \Delta(t^{-1}) d\lambda(t) \right]^{1/p}$$

$$\leq \frac{\varepsilon_2}{M M_o^{1/p}} M_o^{1/p} \left[\int_K \| F^*(t) \|^p d\lambda(t) \right]^{1/p}$$

$$= \frac{\varepsilon_2}{M} \; |||F^*|||_p$$

$$\leq \varepsilon_2 .$$

In view of this inequality, any ε_2-net in the set $\{h*F^*(s): F \in \mathcal{F}\}$, will give a $3\varepsilon_2$-net in $\{F^{**}(s): F \in \mathcal{F}\}$. Now,

$$\begin{aligned}
h_*F^*(s) &= \sum_{i=1}^{n} \alpha_i (\chi_{E_i}\Delta^{-1})*F^*(s) \\
&= \sum_{i=1}^{n} \alpha_i \int_G (\chi_{E_i}\Delta^{-1})(st)\, F^*(t^{-1})\, d\lambda(t) \\
&= \sum_{i=1}^{n} \alpha_i \int_G (\chi_{E_i}\Delta^{-1})(st^{-1})\, F^*(t)\, \Delta(t^{-1})\, d\lambda(t) \\
&= \sum_{i=1}^{n} \alpha_i \Delta^{-1}(s) \int_G \chi_{E_i}(st^{-1})\, F^*(t)\, d\lambda(t) \\
&= \sum_{i=1}^{n} \alpha_i \Delta^{-1}(s) \int_{E_i^{-1}s \cap K} F(t)\, d\lambda(t).
\end{aligned}$$

By (4), the sets $\{\int_{E_i^{-1}s \cap K} F(t)\, d\lambda(t): F \in \mathcal{F}\}$ are relatively compact for $1 \le i \le n$, and hence it follows that the set $\{h_*F^*(s): F \in \mathcal{F}\}$ is relatively compact in X. Thus, we can construct an ε_2-net in this set, and from this we will get a $3\varepsilon_2$-net in $\{F^{**}(s): F \in \mathcal{F}\}$. This proves that $\{F^{**}(s): F^{**} \in \mathcal{F}^{**}\}$ is relatively compact in X, for any $s \in G$.

We note that the family of functions $\mathcal{F}^{**}$ is supported in the compact set UK. Considering $\mathcal{F}^{**}$ as a family of continuous functions from UK into X, we see that this family satisfies the hypothesis of Theorem 7.17 of [20] (Ascoli's theorem). Hence it is relatively compact in the topology of

uniform convergence on UK, i.e. in the supremum norm. Now an $\varepsilon[\lambda(UK)]^{-1/p}$-net in this norm will give an ε-net in F^{**} with the $|||\cdot|||_p$ norm. As we have already proved, this gives a 5ε-net in F. Since $\varepsilon > 0$ is arbitrary, we have proved that F is relatively compact. This completes the proof.

For $X=C$, the complex field, condition (4) is redundant and we get **Weil's theorem** [42]. This is true for finite dimensional spaces also. Condition (4) is important whenever X is infinite dimensional. Indeed, whenever X is infinite dimensional, the following is an example of a family $F \subset L^p(G,X)$ satisfying (1)-(3) but not (4). Take $Y \subset X$ such that Y is bounded but not relatively compact. Take $f \in L^p(G)$, $f \neq 0$. Now define $F = \{fy: y \in Y\}$.

Condition (3) is the left equicontinuity of the functions in F. A similar theorem can be proved with left equicontinuity replaced by right equicontinuity.

<u>Theorem</u> 4.2: A subset F of $L^p(G,X)$ is relatively compact if and only if F satisfies conditions (1), (2) and (4) of Theorem 4.1, and the following condition.

(3)' Given $\varepsilon > 0$, there exists a neighbourhood U of identity e in G such that $\sup\{|||F_a - F|||_p : a \in U, F \in F\} < \varepsilon$.

The proof of Theorem 4.2 is similar to that of Theorem 4.1. One has to take $F^{**} = F * g$ in place of $F^{**} = g * F^*$,

and $\int_G \Delta^{-1} g\, d\lambda = 1$ in place of $\int_G g d\lambda = 1$. Similar changes have to be made in the definition of h. We omit the details.

If we demand both right and left equicontinuity, then we can show that condition (1) follows from the rest. In other words, we shall prove,

Theorem 4.3 A subset F of $L^p(G,X)$ is relatively compact if and only if the following conditions are satisfied.

(1) Given $\varepsilon > 0$, there exists a compact set $K \subset G$ such that $\sup \{ \int_{G-K} |F|^p \, d\lambda : F \in F \} < \varepsilon$.

(2) Given $\varepsilon > 0$, there exists a neighbourhood U of identity e in G, such that $\sup \{ |||_a F - F|||_p, |||F_a - F|||_p : a \in U, F \in F \} < \varepsilon$.

(3) For each measurable relatively compact subset E of G, the set $\{ \int_E F(s)\, d\lambda(s) : F \in F \}$ is relatively compact in X.

Proof. The necessity of the conditions is obvious. For sufficiency, in view of Theorem 4.1, it is enough to prove that (1) - (3) imply that $\sup \{ |||F|||_p : F \in F \} = M < \infty$. For $\varepsilon = 1$, choose a compact set $K \subset G$ as in (1), and a compact neighbourhood U of e in G as in (2). Choose $\{s_i\}_{i=1}^n \subset K$ such that $\{Us_i\}_{i=1}^n$ is a cover of K. Let $F \in F$ and $F'(s) = \frac{1}{\lambda(U)} \int_{sU} F(t)\, d\lambda(t)$. Then $(F'-F)(s)$ $= \frac{1}{\lambda(U)} \int_{sU} (F(t) - F(s))\, d\lambda(t)$. Therefore,

$$\|(F'-F)(s)\| \leq \frac{1}{\lambda(U)} \int_{sU} \|F(t)-F(s)\| \, d\lambda(t)$$

$$= \frac{1}{\lambda(U)} \int_{U} \|F(st)-F(s)\| \, d\lambda(t)$$

$$\leq \left[\frac{1}{\lambda(U)} \int_{U} \|F(st)-F(s)\|^p \, d\lambda(t) \right]^{1/p}.$$

Hence,

$$\int_K |F'-F|^p \, d\lambda \leq \frac{1}{\lambda(U)} \int_K d\lambda(s) \int_U \|F(st)-F(s)\|^p d\lambda(t)$$

$$= \frac{1}{\lambda(U)} \int_U d\lambda(t) \int_K \|F(st)-F(s)\|^p d\lambda(s)$$

$$\leq \frac{1}{\lambda(U)} \int_U \| |F_t-F| \|_p^p \, d\lambda(t)$$

$$\leq 1.$$

Also, for any $a \in U$ and any $s \in G$, we have

$$\|F'(as)-F'(s)\| = \left\| \frac{1}{\lambda(U)} \int_{asU} F(t) d\lambda(t) - \frac{1}{\lambda(U)} \int_{sU} F(t) d\lambda(t) \right\|$$

$$= \left\| \frac{1}{\lambda(U)} \int_{sU} F(at) d\lambda(t) - \frac{1}{\lambda(U)} \int_{sU} F(t) d\lambda(t) \right\|$$

$$\leq \frac{1}{\lambda(U)} \int_{sU} |{}_aF-F| d\lambda$$

$$\leq \left[\frac{1}{\lambda(U)} \int_{sU} |{}_aF-F|^p d\lambda \right]^{1/p}$$

$$\leq \left[\frac{1}{\lambda(U)} \right]^{1/p} = \alpha \quad \text{(say)}.$$

Now $\{ \frac{1}{\lambda(U)} \int_{sU} \phi \, d\lambda : \phi \in F \}$ is relatively compact. Therefore $\sup \{ \| \frac{1}{\lambda(U)} \int_{sU} \phi \, d\lambda \| : \phi \in F \} < \infty$. Let $N = \max_{1 \leq i \leq n} \sup \{ \| \frac{1}{\lambda(U)} \int_{s_iU} \phi \, d\lambda \| : \phi \in F \}$. Then $\|F'(s_i)\| \leq N$ for $i=1,2,\ldots n$.

Since Us_i's cover K, and $||F'(as_i)-F'(s_i)|| \leq \alpha$ for all $a \in U$ and $1 \leq i \leq n$, we have $||F'(s)|| \leq N+\alpha$ for all $s \in K$. Therefore, $\int_K |F'|^p \, d\lambda \leq (N+\alpha)^p \, \lambda(K) = \beta^p$ (say). But $\int_K |F'-F|^p \, d\lambda \leq 1$. Therefore, $\int_K |F|^p \, d\lambda \leq (\beta + 1)^p$. Thus $\int_G |F|^p \, d\lambda \leq (\beta + 1)^p + 1$, and we can take $M = [(\beta+1)^p+1]^{1/p}$. This proves the theorem.

4.3 Continuously Translating Elements of M(G,X):

We now prove that the elements of $L^1(G,X)$ are the only ones in $M(G,X)$, which translate continuously. More precisely, we prove,

Theorem 4.4: If $\mu \in M(G,X)$ is such that $s \to {}_s\mu$ or $s \to \mu_s$ is continuous, then $\mu \in L^1(G,X)$.

Before proving the theorem, we prove a few results. The first result is concerned with convolution of elements of $L^1(G)$ and $M(G,X)$. As discussed in § 2.12, given any $\mu \in M(G,X)$ and $\nu \in M(G)$, we can define $\mu * \nu$ and $\nu * \mu$ as elements of $M(G,X)$, and these satisfy equations (2.1) and (2.3). As promised there, we shall now prove,

Theorem 4.5. If $\mu \in M(G,X)$ and $f \in L^1(G)$, then $\mu * f$ and $f * \mu$ belong to $L^1(G,X)$.

Proof. Since $C_c(G)$ is dense in $L^1(G)$, and $f \to \mu * f$ (or $f * \mu$) is continuous, it is sufficient to prove the result for $f \in C_c(G)$. Let $f \in C_c(G)$. Then f is μ-integrable for any $\mu \in M(G,X)$, and we can define an X-valued function F by,

$$F(s) = \int_G f(t^{-1}s)\, d\mu(t).$$

It is easy to prove that $F \in C_0(G,X)$. Hence F is λ-measurable. Also

$$||F(s)|| \leq \int_G |f(t^{-1}s)|\, dV(\mu)(t)$$

$$\leq V(\mu) * |f|\ (s)$$

Since $V(\mu) * |f| \in L^1(G)$, we see that $F \in L^1(G,X)$. Let ϕ be any element of X^*. Then, for every $E \in \Sigma$,

$$\begin{aligned}
\phi(F(E)) &= \phi\left(\int_E F(s)\, d\lambda(s)\right) \\
&= \int_E (\phi \circ F)(s)\, d\lambda(s) \\
&= \int_E d\lambda(s)\ \phi\left(\int_G f(t^{-1}s)\, d\mu(t)\right) \\
&= \int_E d\lambda(s) \int_G f(t^{-1}s)\, d\,(\phi \circ \mu)(t) \\
&= (\phi \circ \mu) * f(E) \\
&= \phi \circ (\mu * f)(E).
\end{aligned}$$

(see the proof of Proposition 2.16)

Since ϕ is an arbitrary element of X^*, we have $F(E) = \mu * f(E)$ for any $E \in \Sigma$. Therefore, $\mu * f = F \in L^1(G,X)$. The proof that $f * \mu \in L^1(G,X)$ is similar.

<u>Corollary</u>. If $\mu \in M(G,X)$ and $f \in C_c(G)$, then we have the following,

$$\mu * f(s) = \int_G f(t^{-1}s)\, d\mu(t)$$

$$f * \mu(s) = \int_G \Delta(t^{-1})\, f(st^{-1})\, d\mu(t).$$

We can now show that if $F \in L^1(G,X)$ and $f \in L^1(G)$, then the two ways of defining $F * f$ and $f * F$ are equivalent. If $F \in L^1(G,X)$ and $f \in C_c(G)$, then by the corollary above, we have,

$$F * f(s) = \int_G f(t^{-1}s)\, F(t)\, d\lambda(t)$$

We have obtained this by defining $F * f$ through §2.12, after considering F and f to be measures. However, considering F and f to be functions, and using §2.10, we have another definition of $F * f$. According to this definition,

$$F * f(s) = \int_G f(ts)\, F(t^{-1})\, \Delta(t^{-1})\, d\lambda(t)$$

$$= \int_G f(t^{-1}s)\, F(t)\, d\lambda(t).$$

Thus both the definitions coincide, and since $C_c(G)$ is dense in $L^1(G)$, this is true for any $F \in L^1(G,X)$ and $f \in L^1(G)$. For $f * F$, the situation is the same.

We now prove two simple lemmas.

<u>Lemma 4.1</u>. Let $\mu \in M(G,X)$. Then the following are equivalent.

(1) μ is absolutely continuous with respect to λ.

(2) For any measurable, relatively compact set $E \subset G$, the function $s \to \mu(Es)$ is continuous.

(3) For any measurable, relatively compact set $E \subset G$, the function $s \to \mu(sE)$ is continuous.

Proof. Let $\mu \in M(G,X)$, be absolutely continuous with respect to λ, and let E be any measurable, relatively compact subset of G. Let $s_0 \in G$, and let $\varepsilon > 0$ be given. Choose $\delta > 0$, such that for any $E' \subset G$, $\lambda(E') < \delta$ implies $V(\mu)(E') < \varepsilon$. This is possible, since $V(\mu)$ is absolutely continuous with respect to λ. Now $\chi_E \in L^1(G)$, since E is relatively compact. Hence $s \to \chi_{Es}$ is continuous, and we can choose a neighbourhood V of s_0, such that for any $s \in V$, $||\chi_{Es} - \chi_{Es_0}||_1 < \delta$. Therefore, for any $s \in V$

$$\begin{aligned}\lambda(Es \,\Delta\, Es_0) &= \lambda(Es - Es_0) + \lambda(Es_0 - Es)\\ &= ||\chi_{Es} - \chi_{Es_0}||_1\\ &< \delta.\end{aligned}$$

Hence, for any $s \in V$,

$$\begin{aligned}||\mu(Es) - \mu(Es_0)|| &= ||\mu(Es - Es_0) - \mu(Es_0 - Es)||\\ &\leq V(\mu)(Es - Es_0) + V(\mu)(Es_0 - Es)\\ &= V(\mu)(Es \,\Delta\, Es_0)\\ &< \varepsilon .\end{aligned}$$

This shows that $s \to \mu(Es)$ is continuous. This proves (1) $\Longrightarrow$ (2). The proof of (1) $\Longrightarrow$ (3) is similar.

For the proof of (3) ==> (1), let $\mu \in M(G,X)$, such that (3) is satisfied. Let E be any compact subset of G, such that $\lambda(E) = 0$. Then the function $s \to ||\mu(s^{-1}E)||$ is continuous. Let $\nu \in M(G)$ be defined by $d\nu = \chi_U d\lambda$, where U is a relatively compact neighbourhood of e. Then ν is absolutely continuous with respect to λ. Hence $\nu * V(\mu)$ is absolutely continuous with respect to λ. Therefore, we have,

$$\begin{aligned} 0 &= \nu * V(\mu)\,(E) \\ &= \int_G V(\mu)\,(s^{-1}E)\, d\nu(s) \\ &= \int_U V(\mu)\,(s^{-1}E)\, d\lambda(s) \\ &\geq \int_U ||\mu(s^{-1}E)||\, d\lambda(s) \end{aligned}$$

Since $s \to ||\mu(s^{-1}E)||$ is a nonnegative continuous function, $||\mu(s^{-1}E)|| = 0$ for any $s \in$ Interior U. Hence $||\mu(E)|| = 0$. In the same way, $||\mu(E')|| = 0$, for any measurable $E' \subseteq E$. Hence $V(\mu)\ (E) = 0$. This shows that $V(\mu)$ is absolutely continuous with respect to λ. This proves (3) ==> (1). The proof of (2) ==> (1) is similar and the proof of Lemma 4.1 is complete.

Note. The proof of (3) ==> (1), is an adaptation of §19.27 of [13] to the vector-valued case.

Lemma 4.2. Let $\mu \in M(G,X)$ and let E be any measurable,

relatively compact subset of G. Then the functions $s \to \mu(Es)$ and $s \to \mu(sE)$ vanish at inifnity.

Proof. Let $\varepsilon > 0$ be given. By regularity of μ, there exists a compact set $K \subset G$, such that $V(\mu)(K^c) \leq \varepsilon$, where K^c is the complement of K. Let $K_1 = E^{-1}K$. Then K_1 is relatively compact, and for $s \notin K_1$, $Es \subset K^c$. Thus, for $s \notin K_1$, $||\mu(Es)|| \leq \varepsilon$. This shows that $s \to \mu(Es)$ vanishes at infinity. Similarly we can prove that the function $s \to \mu(sE)$ vanishes at infinity, and our proof is complete.

We are now in a position to prove Theorem 4.4.

Proof of Theorem 4.4. Let $\mu \in M(G,X)$, such that $s \to {}_{\bar{s}}\mu$ is continuous. Then for any measurable set $E \subset G$, $s \to {}_{\bar{s}}\mu(E) = \mu(sE)$ is continuous. Hence, by Lemma 4.1 and Lemma 4.2, we can conclude that for any measurable, relatively compact set $E \subset G$, the functions $s \to \mu(sE)$, $s \to \mu(s^{-1}E)$, $s \to \mu(Es)$ and $s \to \mu(Es^{-1})$ are continuous functions vanishing at infinity.

We now take a fixed compact neighbourhood U_0 of the identity e in G. Let $\mathcal{D}$ be the family of all neighbourhoods of e, contained in U_0, directed under inclusion. Take any $W \in \mathcal{D}$. Then $\lambda(W) < \infty$, since W is contained in the compact set U_0. Let $f_W = \frac{1}{\lambda(W)} \chi_W$. Then $f_W \in L^1(G)$, and $||f_W||_1 = 1$. Now, $F_W = \mu * f_W \in M(G,X)$, and $||F_W||_v \leq ||\mu||_v$.

Let $\mathcal{F} = \{F_W : W \in \mathcal{D}\}$. By Theorem 4.5, $\mathcal{F} \subset L^1(G,X)$. We shall now prove that $\mathcal{F}$, as a subset of $L^1(G,X)$, satisfies conditions (1) - (4) of Theorem 4.1

Since for any $W \in \mathcal{D}$, $|||F_W|||_1 = ||F_W||_V \leq ||\mu||_V$, we see that (1) is satisfied with $M = ||\mu||_V$. Next, let $\varepsilon > 0$ be given. Choose a compact set $K_1 \subset G$, such that $V(\mu)\ (K_1^c) < \varepsilon$. Let $K = K_1 U_0$. Then K is compact, and for any $W \in \mathcal{D}$, $\int_{G-K} ||F_W(s)||\,d\lambda(s) = V(F_W)(K^c) \leq V(\mu) * f_W(K^c) \leq \frac{1}{\lambda(W)} \int_W V(\mu)\ (K^c s^{-1})\,d\lambda(s)$. Now for any $s \in K^c$ and for any $t \in W$, $st^{-1} \in K_1^c$. Therefore for any $t \in W$, $K^c t^{-1} \subset K_1^c$ and thus $V(\mu)(K^c t^{-1}) < \varepsilon$. Hence

$$\int_{G-K} ||F_W(s)||\,d\lambda(s) \leq \frac{1}{\lambda(W)} \int_W V(\mu)\ (K^c t^{-1})\,d\lambda(t) \leq \frac{\varepsilon}{\lambda(W)} \int_W d\lambda(t) = \varepsilon.$$

Thus $\mathcal{F}$ satisfies condition (2) of Theorem 4.1.

Again for $\varepsilon > 0$, we take a neighbourhood U of e in G, such that for any $s \in U$, $||{}_s\mu - \mu||_V \leq \varepsilon$. This is possible since $s \to {}_s\mu$ is continuous. Then for any $W \in \mathcal{D}$ and for any

$$s \in U, |||{}_sF_W - F_W|||_1 = ||{}_s(\mu * F_W) - \mu * f_W||_V = ||{}_s\mu * f_W$$

$$- \mu * f_W||_V = ||({}_s\mu - \mu) * f_W||_V \leq ||{}_s\mu - \mu||_V ||f_W||_1 \leq \varepsilon .$$

Thus $\mathcal{F}$ satisfies condition (3) of Theorem 4.1.

Finally, let E be any measurable, relatively compact subset of G. We shall show that $\{F_W(E) : W \in \mathcal{D}\}$ is relatively

compact in X. First we note that since E is relatively compact, the function $s \to \mu(Es^{-1})$ is a continuous function vanishing at infinity. Thus this function has separable range. Hence by Proposition 2.16, equation (2.2) is valid for μ. Thus

$$F_W(E) = \int_G \mu(Et^{-1})df_W(t) = \frac{1}{\lambda(W)} \int_W \mu(Et^{-1})d\lambda(t).$$

Since $t \to \mu(Et^{-1})$ is continuous, and U_0 is compact, the function $t \to \mu(Et^{-1})$ is uniformly continuous on U_0, i.e., given $\varepsilon > 0$, there exists a neighbourhood W_0 of e, such that for any $s,t \in U_0$ with $st^{-1} \in W_0$, $||\mu(Es^{-1}) - \mu(Et^{-1})|| \leq \varepsilon$. Cover U_0 with finite number of right translates of W_0, $\{W_0 s_i\}_{i=1}^{n}$. Then any $W \in \mathcal{D}$ can be expressed as $W = \bigcup_{i=1}^{m} W_i$, where W_i's are disjoint measurable sets and each $W_i \subset W_0 s_{k_i}$ for some $1 \leq k_i \leq n$. Now if $s \in W_i$ then $s \in W_0 x_{k_i}$ and hence $ss_{k_i}^{-1} \in W_0$. Thus, for $s \in W_i$, $||\mu(Es_{k_i}^{-1}) - \mu(Es^{-1})|| \leq \varepsilon$. Therefore,

$$||F_W(E) - \sum_{i=1}^{m} \frac{\lambda(W_i)\ \mu(Es_{k_i}^{-1})}{\lambda(W)}||$$

$$= \frac{1}{\lambda(W)} ||\sum_{i=1}^{m} \int_{W_i} \mu(Es^{-1})d\lambda(s) - \sum_{i=1}^{m} \lambda(W_i)\mu(Es_{k_i}^{-1})||$$

$$\leq \frac{1}{\lambda(W)} \sum_{i=1}^{m} \int_{W_i} ||\mu(Es^{-1}) - \mu(Es_{k_i}^{-1})||\ d\lambda(s)$$

$$\leq \frac{\varepsilon}{\lambda(W)} \sum_{i=1}^{m} \int_{W_i} d\lambda(s) = \varepsilon .$$

Let Y be the finite dimensional linear space generated by $\{\mu(Es_j^{-1})\}_{j=1}^{n}$. Then we see that for any $W \in \mathcal{D}$, there exists $a_W \in Y$, such that $||F_W(E) - a_W|| \leq \varepsilon$. Since $\{F_W(E): W \in \mathcal{D}\}$ is bounded, $\{a_W: W \in \mathcal{D}\}$ is a bounded subset of the finite dimensional linear space Y. Hence $\{a_W: W \in \mathcal{D}\}$ is totally bounded, and we can obtain an ε-net $\{a_{W_{(i)}}\}_{i=1}^{m}$ in $\{a_W: W \in \mathcal{D}\}$. Then it is easy to see that $\{F_{W_{(i)}}(E)\}_{i=1}^{m}$ is a 3ε-net in $\{F_W(E): W \in \mathcal{D}\}$. Since ε is arbitrary, we can conclude that $\{F_W(E): W \in \mathcal{D}\}$ is totally bounded and hence relatively compact in X. This shows that $\mathcal{F}$ satisfies condition (4) of Theorem 4.1.

Thus $\mathcal{F}$ satisfies all the conditions of Theorem 4.1 and hence $\mathcal{F}$ is relatively compact as a subset of $L^1(G,X)$. Therefore the net $\{F_W: W \in \mathcal{D}\}$ has a subnet which converges to some $F \in L^1(G,X)$. Let E be any measurable relatively compact subset of G. Then the corresponding subnet of $\{F_W(E): W \in \mathcal{D}\}$ converges to $F(E)$. However, since E is relatively compact, we have $F_W(E) = \frac{1}{\lambda(W)} \int_W \mu(Et^{-1})d\lambda(t)$, as has been already shown. Since E is relatively compact $t \to \mu(Et^{-1})$ is continuous. Hence, given $\varepsilon > 0$, we can choose a neighbourhood U of e in G such that for any $t \in U$, $||\mu(Et^{-1}) - \mu(E)|| \leq \varepsilon$. Then for $W \subset U$ and for $t \in W$, $||\mu(Et^{-1}) - \mu(E)|| \leq \varepsilon$. Therefore, for $W \in \mathcal{D}$ and $W \subset U$,

$$||F_W(E)-\mu(E)|| \leq ||\frac{1}{\lambda(W)} \int_W \{\mu(Et^{-1})-\mu(E)\}d\lambda(t)||$$

$$\leq \frac{1}{\lambda(W)} \int_W ||\mu(Et^{-1})-\mu(E)||d\lambda(t)$$

$$\leq \frac{\varepsilon}{\lambda(W)} \int_W d\lambda(t) = \varepsilon .$$

Therefore, the net $\{F_W(E): W \in \mathcal{D}\}$ converges to $\mu(E)$. Hence any subnet of it also converges to $\mu(E)$, and thus $\mu(E) = F(E)$ for any measurable, relatively compact subset E of G. Since μ and F are regular this equality remains valid for all measurable subsets E of G. Thus $\mu = F \in L^1(G,X)$. This completes the proof of one-half of the theorem.

For the proof of the other half of the theorem, let $\mu \in M(G,X)$ such that $s \to \mu_s$ is continuous. Then, for any measurable set $E \subset G$, $s \to \mu_s(E) = \Delta(s^{-1}) \mu(Es)$ is continuous. However $s \to \Delta(s)$ is continuous. Therefore, $s \to \mu(Es)$ is continuous. Hence, by Lemma 4.1 and Lemma 4.2, we can conclude that for any measurable relatively compact set E, the functions $s \to \mu(sE)$, $s \to \mu(s^{-1}E)$, $s \to \mu(Es)$ and $s \to \mu(Es^{-1})$ are continuous functions vanishing at infinity.

The rest of the proof is similar to that of the first half of the theorem. Instead of $F_W = \mu * f_W$, we shall have to take $F_W = f_W * \mu$. Theorem 4.2, instead of Theorem 4.1, will be used to prove that F is relatively compact in $L^1(G,X)$, and as before we will be able to conclude that $\mu \in L^1(G,X)$. This completes the proof.

bounded approximate identity, we can use the arguments of (3.3) of [32], to show that this map is injective. Considering the net $\{f_W\} \subset L^1(G)$, as defined in the proof of Theorem 4.4, it is easy to see that $\{f_W * F\}$ converges to F for every $F \in C_0(G,X)$. This shows that elements of the form $f*F$ for $f \in L^1(G)$ and $F \in C_0(G,X)$, are dense in $C_0(G,X)$. Hence, by the module factorisation theorem (§32.22 of [14]), we have that any $F \in C_0(G,X)$ can be expressed as $f*F'$ with $f \in L^1(G)$ and $F' \in C_0(G,X)$. Thus, the above map is surjective. For $f \in C_0(G,X)$, let $|||F|||_{\otimes}$ denote the norm of the preimage of F under the above map. We already have, $|||F|||_{\infty} \leq |||F|||_{\otimes}$. Hence, the lemma will be proved if we prove $|||F|||_{\otimes} \leq |||F|||_{\infty}$. Taking the net $\{f_W\}$ as considered above, we have $f_W * F \to F$ in the $|||\cdot|||_{\infty}$ norm. However, by the open-mapping theorem, $|||\cdot|||_{\otimes}$ is equivalent to $|||\cdot|||_{\infty}$. Therefore, $|||f_W * F|||_{\otimes} \to |||F|||_{\otimes}$. But $|||f_W * F|||_{\otimes} \leq ||f_W||_1 |||F|||_{\infty} = |||F|||_{\infty}$. Hence $|||F|||_{\otimes} \leq |||F|||_{\infty}$, and the lemma is proved.

Lemma 5.2. Let $f \in L^1(G)$, $F \in C_0(G,X)$ and $\mu \in M(G,X^*)$. Then

$$\langle \mu * f, F \rangle = \langle \mu, f*F \rangle .$$

Proof. The lemma is true for the scalar-valued case, where a simple application of the Fubini theorem gives the result.

Now let $x \in X$. Then $x \circ \mu \in M(G)$, and we have, $x \circ (\mu * f) = (x \circ \mu) * f$ (see the proof of Proposition 2.16). Let $g \in C_0(G)$. If $F = gx$, then

$$\begin{aligned}
\langle \mu * f, F \rangle &= \int_G F(-s)\, d(\mu * f)(s) \\
&= x\left(\int_G g(-s)\, d(\mu * f)(s) \right) \\
&= \int_G g(-s)\, d(x \circ (\mu * f))(s) \\
&= \int_G g(-s)\, d((x \circ \mu) * f)(s) \\
&= \langle (x \circ \mu) * f, g \rangle .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \mu, f * F \rangle &= x\left(\int_G f * g(-s)\, d\mu(s) \right) \\
&= \int_G f * g(-s)\, d(x \circ \mu)(s) \\
&= \langle x \circ \mu , f * g \rangle
\end{aligned}$$

Since, the lemma is true for the scalar-valued case, we have

$$\langle (x \circ \mu) * f, g \rangle = \langle x \circ \mu , f * g \rangle$$

Hence,

$$\langle \mu * f, F \rangle = \langle \mu , f * F \rangle$$

By linearity, this is true for any F of the form $\sum_{i=1}^{n} g_i x_i$, with each $g_i \in C_0(S)$ and $x_i \in X$. Since functions of this form are dense in $C_0(G,X)$, the result is true for any $F \in C_0(G,X)$, and our lemma is proved.

It is well-known that the convolution of two elements $f,g \in L^1(G)$, can be expressed as a $L^1(G)$-valued vector integral as follows (see §6.6 of Chapter 3 of [28]).

$$f * g = g * f = \int_G (g)_{-s}\, f(s)\, d\lambda(s) \qquad (5.1)$$

A similar result can be obtained for $F \in L^1(G,X)$ and $g \in L^1(G)$. Since $s \to F_s$ is continuous, it is easy to prove the integral $\int (F)_{-s}\, g(s)\, d\lambda(s)$ exists as an element of $L^1(G,X)$. Let $\phi \in X^*$. Then $\phi o F \in L^1(G)$, and it is easy to prove that $\phi o(F * g) = (\phi o F) * g$ and $\phi o(g * F) = g * (\phi o F)$. Now, since $\phi o F \in L^1(G)$, we have,

$$\begin{aligned}(\phi o F) * g = g * (\phi o F) &= \int_G (\phi o F)_{-s} g(s)\, d\lambda(s) \\ &= \int_G \phi o(F_{-s})\, g(s)\, d\lambda(s) \\ &= \phi\Big(\int_G (F)_{-s} g(s)\, d\lambda(s)\Big)\end{aligned}$$

Hence,

$$\phi o(F * g) = \phi o(g * F) = \phi \cdot\Big(\int_G (F)_{-s}\, g(s)\, d\lambda(s)\Big).$$

Since $\phi \in X^*$ is arbitrary, we have proved,

<u>Lemma 5.3</u>. If $F \in L^1(G,X)$ and $g \in L^1(G)$, then $F * g = g * F = \int_G (F)_{-s}\, g(s)\, d\lambda(s)$.

This lemma allows us to identify the invariant operators from $L^1(G)$ into $L^1(G,X)$, with the $L^1(G)$-module homomorphisms.

Lemma 5.4. Let T be an invariant bounded linear operator from $L^1(G)$ into $L^1(G,X)$. Then $T \in \mathrm{Hom}_G(L^1(G), L^1(G,X))$, i.e. $T(f * g) = f * T(g)$, for any $f,g \in L^1(G)$.

Proof. Let $f,g \in L^1(G)$. Then using equation (5.1)

$$T(f * g) = T(\int_G (g)_{-s} f(s)\, d\lambda(s))$$

$$= \int_G (T \circ g_{-s})\, f(s)\, d\lambda(s)$$

$$= \int_G (T \circ g)_{-s}\, f(s)\, d\lambda(s) \qquad \text{(since } T \text{ is invariant)}$$

$$= f * T(g).$$

This proves the lemma.

Similarly we can prove,

Lemma 5.5. Let T be an invariant bounded linear operator of $L^1(G,X)$. Then $T \in \mathrm{Hom}_G\,(L^1(G,X))$.

5.2 The space $\mathrm{Hom}_G\,(L^1(G), L^1(G,X))$: We shall now prove that $\mathrm{Hom}_G\,(L^1(G), L^1(G,X))$ is isometrically isomorphic to $M(G,X)$. If $f \in L^1(G)$ and $\mu \in M(G,X)$, then considering f as an element of $M(G)$, $\mu * f$ is defined as an element of $M(G,X)$. By Theorem 4.5, $\mu * f \in L^1(G,X)$. Since $(\mu * f)_s = \mu * f_s$ for any $s \in G$, it follows that $f \to \mu * f$ is a bounded linear invariant operator from $L^1(G)$ into $L^1(G,X)$. By Lemma 5.4, this defines an element of $\mathrm{Hom}_G(L^1(G), L^1(G,X))$. We now prove,

Theorem 5.1. Let $T \in \mathrm{Hom}_G\,(L^1(G), L^1(G,X))$. Then there exists a $\mu \in M(G,X)$, such that $T(f) = \mu * f$ for any $f \in L^1(G)$.

Moreover this correspondence gives an isometric isomorphism between $\mathrm{Hom}_G(L^1(G), L^1(G,X))$ and $M(G,X)$.

Proof. Let $T \in \mathrm{Hom}_G(L^1(G), L^1(G,X))$. Considering the natural isometric imbedding of X in its second dual X^{**}, $L^1(G,X)$ can be imbedded isometrically in $M(G,X^{**})$. Hence T can be considered as an element of $\mathrm{Hom}_G(L^1(G), M(G,X^{**}))$ with the same norm. However, $M(G,X^{**})$ is isometrically isomorphic to $C_o(G,X^*)^*$ by Theorem 2.1. Hence, by Rieffel's result on module homomorphisms (see §2.14), there exists $\Phi \in (L^1(G) \otimes_G C_o(G,X^*))^*$ with $||\Phi|| = ||T||$, such that

$$\langle \Phi, f \otimes F \rangle = \langle T(f), F \rangle$$

for any $F \in C_o(G,X^*)$ and $f \in L^1(G)$. Here, $T(f)$ has been considered as an element of $M(G,X^{**})$ which is the dual of $C_o(G,X^*)$. However, by Lemma 5.1, $L^1(G) \otimes_G C_o(G,X^*)$ is isometrically isomorphic to $C_o(G,X^*)$ under the map $f \otimes F \to f * F$. Hence, there exists $\mu \in M(G,X^{**})$ with $||\mu||_v = ||T||$, such that

$$\langle \mu, f * F \rangle = \langle T(f), F \rangle$$

for any $F \in C_o(G,X^*)$ and $f \in L^1(G)$. Hence by Lemma 5.2, we have

$$\langle T(f), F \rangle = \langle \mu * f, F \rangle$$

for any $F \in C_o(G,X^*)$. Therefore, $T(f) = \mu * f$, for any $f \in L^1(G)$. The association of μ with T is obviously

linear, and since $||\mu||_v = ||T||$, the proof of the theorem will be complete if we prove that μ takes values only in X.

Let us take the canonical map ψ from X^{**} onto the quotient space $Y = X^{**}/X$. Consider the Y-valued measure $\nu = \psi \circ \mu$. Obviously $\nu \in M(G,Y)$ and $\nu * f = \psi \circ (\mu * f)$, for any $f \in L^1(G)$. However $\mu * f = T(f) \in L^1(G,X)$, and hence for any $E \in \Sigma$, $\mu * f(E) \in X$. This means that $\nu * f = 0$ for any $f \in L^1(G)$. Consider ν as an element of $M(G,Y^{**})$ which is the dual of $C_0(G,Y^*)$. Take any $F \in C_0(G,Y^*)$. Then $F = f * F'$ for some $f \in L^1(G)$ and $F' \in C_0(G,Y^*)$ (see the proof of Lemma 5.1). Therefore,

$$\begin{aligned} \langle \nu, F \rangle &= \langle \nu, f * F' \rangle \\ &= \langle \nu * f, F' \rangle \\ &= 0 \end{aligned}$$

Hence $\nu = 0$, i.e. $\psi \circ (\mu(E)) = 0$ for any $E \in \Sigma$. This shows that $\mu(E) \in X$ for any $E \in \Sigma$, and our proof is complete.

Corollary: $\mathrm{Hom}_G\,(L^1(G),\, L^1(G,X))$ is precisely the set of invariant bounded linear operators from $L^1(G)$ into $L^1(G,X)$.

5.3 Invariant Operators of $L^1(G,X)$: We now characterise the invariant, bounded linear operators of $L^1(G,X)$. Let T be such an operator. Then, by Lemma 5.5, $T \in \mathrm{Hom}_G(L^1(G,X))$. Take any $x \in X$, and define a mapping T_x from $L^1(G)$ into

$L^1(G,X)$ by $T_x(f) = T(fx)$. It is easy to prove that $T_x \in \mathrm{Hom}_G(L^1(G), L^1(G,X))$. Also $||T_x|| \leq ||x||\ ||T||$. Therefore, by Theorem 5.1, there exists a measure $M(x) \in M(G,X)$, such that $T_x(f) = M(x) * f$, and $||M(x)||_V \leq ||x||\ ||T||$. It is easy to see that $x \to M(x)$ is linear, and hence from T we have obtained a bounded linear operator M from X into $M(G,X)$ with $||M|| \leq ||T||$, such that $T(fx) = M(x) * f$, for all $x \in X$ and $f \in L^1(G)$.

Conversely, let M be a bounded linear operator from X into $M(G,X)$. Then $(f,x) \to M(x) * f$ gives a bilinear map from $L^1(G) \times X$ into $L^1(G,X)$, such that $|||M(x) * f|||_1 = ||M(x) * f||_V \leq ||M(x)||_V\ ||f||_1 \leq ||M||\ ||x||\ ||f||_1$. Hence, by the universal property of tensor products with respect to bounded bilinear maps (see §2.5), we get a bounded linear map T' from $L^1(G) \otimes_\gamma X$ into $L^1(G,X)$ with $||T'|| \leq ||M||$, such that $T'(f \otimes x) = M(x) * f$ for any $f \in L^1(G)$ and $x \in X$. However, $L^1(G) \otimes_\gamma X$ is isometrically isomorphic to $L^1(G,X)$. Hence we get a bounded linear operator T of $L^1(G,X)$ with $||T|| \leq ||M||$, such that $T(fx) = M(x) * f$, for any $f \in L^1(G)$ and $x \in X$. Let $s \in S$. Then $T((fx)_s) = T(f_s x) = M(x) * f_s = (M(x) * f)_s = (T(fx))_s$. Since functions of the form $\sum_{i=1}^{n} f_i x_i$ with $f_i \in L^1(G)$ and $x_i \in X$, are dense in $L^1(G,X)$, and since translation is continuous in $L^1(G,X)$, it follows that T is invariant.

By the previous paragraph, we can obtain a bounded linear transformation from X into $M(G,X)$, associated with T. It is easy to see that this transformation is nothing but M, and hence $||M|| \leq ||T||$. Therefore $||T|| = ||M||$, and it is easy to see that this association is linear. Hence we have proved,

<u>Theorem 5.2</u>. $\mathrm{Hom}_G(L^1(G,X))$ is precisely the set of bounded linear invariant operators of $L^1(G,X)$, and this space is isometrically isomorphic to $\mathcal{L}(X,M(G,X))$, the space of bounded linear operators from X into $M(G,X)$.

5.4 <u>Multipliers of $L^1(G,A)$</u>: Let A be a commutative, semisimple Banach algebra with an identity e, such that $||e|| = 1$. We first prove a couple of lemmas.

<u>Lemma 5.6</u>. A bounded linear operator of $L^1(G,A)$ is a multiplier if and only if T is invariant and $T \in \mathrm{Hom}_A(L^1(G,A))$.

<u>Proof</u>. Let T be a multiplier of $L^1(G,A)$. Then for any $f \in L^1(G)$ and $F \in L^1(G,A)$,

$$\begin{aligned} T(f * F) &= T(fe * F) \\ &= fe * T(F) \\ &= f * T(F) \end{aligned}$$

Therefore $T \in \mathrm{Hom}_G(L^1(G,A))$, and hence T is invariant. Next, take the net $\{f_W\} \subset L^1(G)$ as defined in the proof

of Theorem 4.4. Let $F \in L^1(G,A)$. Using the continuity of $s \to \bar{F}_s$, it is easy to prove that $f_W * F \to F$. Take any $a \in A$, and $F \in L^1(G,A)$. Then

$$
\begin{aligned}
T(aF) &= T\left[\lim_W f_W * (aF)\right] \\
&= \lim_W T(f_W * (aF)) \\
&= \lim_W T((f_W a) * F) \\
&= \lim_W \left[(f_W a) * T(F)\right] \\
&= \lim_W (f_W * aT(F)) \\
&= a\,T(F).
\end{aligned}
$$

Hence, $T \in \mathrm{Hom}_A\,(L^1(G,A))$.

Conversely, if T is invariant (i.e. $T \in \mathrm{Hom}_G(L^1(G,A))$, and $T \in \mathrm{Hom}_A(L^1(G,A))$, then for any $a \in A$, $f \in L^1(G)$ and $\bar{F}' \in L^1(G,A)$,

$$
\begin{aligned}
T((fa) * \bar{F}') &= T(a(f * \bar{F}')) \\
&= a\,T(f * \bar{F}') \\
&= a\left[f * T(\bar{F}')\right] \\
&= (fa) * T(\bar{F}')
\end{aligned}
$$

Since linear combinations of the functions of the form 'fa' are dense in $L^1(G,A)$, we have $T(F * \bar{F}') = F * T(\bar{F}')$ for any $F, \bar{F}' \in L^1(G,A)$. This completes the proof.

The next lemma is similar to Theorem 4.5.

Lemma 5.7. If $\mu \in M(G,A)$ and $F \in L^1(G,A)$, then $\mu * F \in L^1(G,A)$.

Proof. Let us take any F of the form $F = fa$ for $a \in A$ and $f \in L^1(G)$. Then, $\mu * F = \mu * (fa) = a(\mu * f)$. But by Theorem 4.5, $\mu * f \in L^1(G,A)$. Therefore $\mu * (fa) \in L^1(G,A)$. Since linear combinations of functions of this form are dense in $L^1(G,A)$, it follows that $\mu * F \in L^1(G,A)$ for any $F \in L^1(G,A)$. This completes the proof.

Let $\mu \in M(G,A)$ and consider the bounded linear map T from $L^1(G,A)$ into itself defined by $T(F) = \mu * F$. Since $(\mu * F)_s = \mu * F_s$ for any $s \in G$, it follows that T is invariant. Also since $\mu * (aF) = a(\mu * F)$, $T \in \mathrm{Hom}_A(L^1(G,A))$. Hence, by Lemma 5.6, T is a multiplier of $L^1(G,A)$. In the following theorem, we prove that all the multipliers of $L^1(G,A)$ are given by elements of $M(G,A)$ in this way.

Theorem 5.3. Let T be a multiplier of $L^1(G,A)$. Then there exists a $\mu \in M(G,A)$ such that $T(F) = \mu * F$ for all $F \in L^1(G,A)$. Moreover, this correspondence gives an isometric isomorphism between $\mathrm{Hom}_{L^1(G,A)}(L^1(G,A))$ and $M(G,A)$.

Proof. Let T be a multiplier of $L^1(G,A)$. Then by Lemma 5.6, $T \in \mathrm{Hom}_G(L^1(G,A))$. Therefore, by Theorem 5.2, there exists $M \in \mathcal{L}(A,M(G,A))$ with $||M|| = ||T||$, such

that $T(fa) = M(a) * f$, for any $a \in A$ and $f \in L^1(G)$. By Lemma 5.6 again, $T \in Hom_A(L^1(G,A))$. Therefore,

$$
\begin{aligned}
T(fa) &= T(a(fe)) \\
&= a\,T(fe) \\
&= a\,(M(e) * f) \\
&= M(e) * (fa)
\end{aligned}
$$

Since, linear combinations of functions of the form 'fa' are dense in $L^1(G,A)$, we have for any $F \in L^1(G,A)$, $T(F) = \mu * F$, where $\mu = M(e)$. Now, $||\mu||_v \leq ||M||\ ||e|| = ||T||$. On the otherhand, $|||T(F)|||_1 \leq ||\mu||_v\ |||F|||_1$. Therefore $||T|| \leq ||\mu||_v$. This proves $||\mu||_v = ||T||$, and obviously this association is linear. This completes the proof.

CHAPTER VI

SOME RESULTS ON MULTIPLIERS

6.1 In this chapter, we shall derive some results on multipliers of any regular, commutative, semisimple, Tauberian Banach algebra A satisfying the following condition.

> There exists a constant $K > 0$, such that for each neighbourhood V of any element m of the maximal-
>
> * ideal space M of A, there exists $f \in A$ with Gelfand transform $\hat{f}$ supported by V, such that $\hat{f}(m) = 1$, and $||f|| \leq K/||m||_{A^*}$.

We shall see that any Segal algebra on a locally compact abelian group G, in particular $L^1(G)$, satisfies this condition. We note that there are regular, commutative, semisimple, Tauberian Banach algebras which do not satisfy this condition. For example, we can take the space $C^1[0,1]$ of continuously differentiable functions on $[0,1]$, which forms a Banach algebra under point-wise product with the norm $||f|| = ||f||_\infty + ||f'||_\infty$, where f' is the derivative of f. We now investigate whether the vector-valued group algebra $L^1(G,A)$ satisfies the condition * . We prove,

<u>Theorem 6.1</u>: Let A be a regular, commutative, semisimple, Tauberian Banach algebra, and let G be a locally compact

abelian group. Then $L^1(G,A)$ satisfies the condition * if and only if A satisfies *.

Proof. Let M be the maximal-ideal space of A, and let Γ be the dual of G. Then $\Gamma \times M$ is the maximal-ideal space of $L^1(G,A)$, and $||(\gamma,m)||_{L^1(G,A)^*} = ||m||_{A^*}$ (see §2.11). Let A satisfy *, and let V be an arbitrary neighbourhood of an arbitrary element $(\gamma,m) \in \Gamma \times M$. Then we can find neighbourhoods W_1 and W_2, of γ and m respectively, such that $W_1 \times W_2 \subset V$. Choose $f \in L^1(G)$, such that $\hat{f}(\gamma) = 1$, $\hat{f}$ is supported in W_1 and $||f||_1 = 1$. This is possible by 2.6.1 of [34]. Since A satisfies *, we can find $a \in A$, such that $\hat{a}(m) = 1$, $\hat{a}$ is supported in W_2 and $||a|| \leq K/||m||_{A^*} = K/||(\gamma,m)||_{L^1(G,A)^*}$. Taking $F = fa$, we see that $|||F|||_1 \leq K/||(\gamma,m)||_{L^1(G,A)^*}$, $\hat{F}((\gamma,m)) = \hat{a}(m)\,\hat{f}(\gamma) = 1$, and $\hat{F}$ is supported in $W_1 \times W_2 \subset V$. This proves that $L^1(G,A)$ satisfies *.

Conversely, let $L^1(G,A)$ satisfy *. Let W be an arbitrary neighbourhood of an arbitrary element $m \in M$. Then $V = \Gamma \times W$ is a neighbourhood of $(1,m)$ in $\Gamma \times M$. Since $L^1(G,A)$ satisfies *, we can find $F \in L^1(G,A)$ such that $\hat{F}((1,m)) = 1$, $\hat{F}$ is supported in V, and $|||F|||_1 \leq K/||(1,m)||_{L^1(G,A)^*}$. Let $a = \int_G F d\lambda$. Then $a \in A$, $\hat{a}(m) = m(\int_G F d\lambda) = (1,m)(F) = 1$. Also, $\hat{a}$ is supported in W

and $||a|| = || \int_G F d\lambda || \leq |||F|||_1 \leq K/||(1,m)||_{L^1(G,A)^*} = K/||m||_{A^*}$. This proves that A satisfies $*$, and the the theorem is proved.

6.2 The main results: We now prove the main results of this chapter. Let A be a commutative, semisimple Banach algebra. Let T be a multiplier of the form $T = T_1 T_2$, where T_1 is an invertible multiplier, and T_2 is an idempotent multiplier. Then it is not difficult to see that $T^2(A)$ is closed. Indeed, if T is of the given form, then we first note that $T_2(A)$ is closed. To see this, let $f_n \in T_2(A)$, for $n = 1,2,3,\ldots$; and let $f_n \to f$. Then, since T_2 is an idempotent $T_2(f_n) = f_n$. Hence $T_2(f) = f$, which shows that $T_2(A)$ is closed. Now, since T_1 has a continuous inverse, it maps closed sets to closed sets. Therefore, $T(A) = T_1(T_2(A))$ is closed. Now, since the multipliers commute with one another, we have $T^2 = T_1^2 T_2^2$. Since T_1 is invertible, T_1^2 is also invertible. Also $T_2^2 = T_2$ is an idempotent. Therefore $T^2(A)$ is also closed. Thus, if T is the product of an idempotent and an invertible multiplier, then $T^2(A)$ is closed. In what follows, we try to prove a converse of this fact for certain special commutative Banach algebras.

Theorem 6.2. Let T be a multiplier of a regular, commutative, semisimple, Tauberian Banach algebra A satisfying the condition $*$. Then $T^2(A)$ is closed if and only if T is the product of an idempotent and an invertible multiplier.

Before proving Theorem 6.2 we prove a few lemmas.

Lemma 6.1. Let T be a multiplier of a Banach algebra A as in Theorem 6.2. If I is an ideal in A, such that $T(I)$ is closed, then $\hat{T}$ is bounded away from zero on $W = M - \text{hull } T(I) = M - (\hat{T}^{-1}(0) \cup \text{hull } I)$.

Proof. Since T is continuous, we can take I to be closed without loss of generality. Now, by the open-mapping theorem, there exists $K_1 > 0$, such that for any $f \in T(I)$, we can find $g \in I$, such that $f = T(g)$ and $||g|| \leq K_1||f||$.

Let $m \in W$. Now W is open, and hence we can choose a compact neighbourhood V of m contained in W. Since A satisfies $*$, there exists $f \in A$, such that $\hat{f}$ is supported in V, $\hat{f}(m) = 1$ and $||f|| \leq K||m||_{A^*}$, where K is a constant independent of m and V. Since T is a multiplier, $T(I)$ is an ideal. Also, it is given to be closed. Now, since $\hat{f}$ has compact support disjoint from hull $T(I)$, regularity of A implies $f \in T(I)$. Hence we can find $g \in I$, such that $f = T(g)$ and $||g|| \leq K_1||f|| \leq KK_1/||m||_{A^*}$. Therefore, $1 = \hat{f}(m) = \hat{T}(m)\,\hat{g}(m) = |\hat{T}(m)\hat{g}(m)| \leq |\hat{T}(m)|\ ||m||_{A^*}\ ||g|| \leq |\hat{T}(m)|\ KK_1$. Therefore, $|\hat{T}(m)| \geq 1/KK_1$. Since m is an arbitrary element of W, we can conclude that $\hat{T}$ is bounded away from zero on W. This completes the proof.

<u>Lemma 6.2</u>. Let A be any commutative, semisimple Banach algebra. Let T, T_1 be multipliers of A such that $\hat{T}_1(m) = (\hat{T}(m))^{-1}$ for all $m \in M$ satisfying $\hat{T}(m) \neq 0$. Then T is the product of an idempotent and an invertible multiplier.

<u>Proof</u>. Let $T_2 = T_1^2\, T$, $E = T_1 T$ and $K = \hat{T}^{-1}(0)$. Then we see that

$$\hat{T}_2(m) = \begin{cases} 0 & \text{on } K \\ (\hat{T}(m))^{-1} & \text{outside } K \end{cases}$$

and $\hat{E} = \chi_{K^c}$, where K^c is the complement of K. Thus E is an idempotent. Let $T' = T+I-E$, where I is the identity operator. It is easy to check that $T_2 T = T_1^2\, T^2 = E^2 = E$, $T_2 E = T_2$ and $TE = T$. Therefore,

$$\begin{aligned} T'(T_2+I-E) &= (T+I-E)\ (T_2+I-E) \\ &= E + T_2 - T_2 + T - T + I - E \\ &= I. \end{aligned}$$

Hence T' is invertible. Finally, we note that $T = ET = E(T+I-E) = ET'$. Thus T is the product of an idempotent and an invertible multiplier. This completes the proof.

<u>Proof of Theorem 6.2</u>. We have already noted the proof of the 'if' part. For the 'only if' part, let T be a multiplier such that $T^2(A)$ is closed. Let $K = \hat{T}^{-1}(0)$. Then hull $T^2(A)$ = hull $T(A) = K$. Now $T(A)$ is an ideal and

$T(T(A)) = T^2(A)$ is closed. Hence, by Lemma 6.1, $\hat{T}$ is bounded away from zero on K^c, the complement of K. Hence K is open and closed. Since A is regular and Tauberian, K is a set of spectral synthesis, i.e. the only closed ideal with hull K is $k(K) = \{f \in A: \hat{f} = 0 \text{ on } K\}$. Hence $T^2(A) = k(K)$. But $T^2(A) \subset T(A) \subset k(K)$. Therefore $T^2(A) = T(A) = k(K)$. Also if $f \in k(K)$ and $T(f)=0$, then $\hat{f}$ is identically zero. Since A is semisimple, we get $f = 0$. Hence T', the restriction of T to $k(K)$ is a continuous bijection of $k(K)$ onto $k(K)$. Therefore $T'^{-1} = T_o$ is continuous. Also, for all $f \in k(K)$

$$\widehat{T_o(f)}(m) = \begin{cases} (\hat{T}(m))^{-1}\, \hat{f}(m) & \text{on } K^c \\ 0 & \text{on } K. \end{cases}$$

Consider $T_1 = T_o^2\, T$. T_1 is a bounded linear map from A into A, such that

$$\widehat{T_1(f)}(m) = \begin{cases} (\hat{T}(m))^{-1}\, \hat{f}(m) & \text{on } K^c \\ 0 & \text{on } K. \end{cases}$$

Thus T_1 is a multiplier and T_1, T satisfy the hypothesis of Lemma 6.2. Therefore T is the product of an idempotent and an invertible multiplier, and the proof is complete.

We have seen that if T is the product of an idempotent and an invertible multiplier, then T(A) is closed. What

about the converse of this fact? Lemmas 6.1 and 6.2 can be used to provide partial answers to this question.

<u>Theorem 6.3</u>. Let T be a multiplier of a commutative Banach algebra A as in Theorem 6.2, whose maximal-ideal space $\mathcal{M}$ is connected. Then $T(A)$ is closed if and only if either $T = 0$, or T is invertible.

<u>Proof</u>. The 'if' part is obvious. For the 'only if' part, let $K = \hat{T}^{-1}(0)$. If $T(A)$ is closed, by Lemma 6.1, $\hat{T}$ is bounded away from zero on K^c. Hence K is open and closed. Since $\mathcal{M}$ is connected, either $K = \mathcal{M}$ or K is empty. In the former case, $\hat{T}$ is identically zero, and hence $T = 0$. In the latter case, hull $T(A)$ is empty, and hence $T(A) = A$. Also if $T(f) = 0$, then $\hat{f} = 0$, and hence $f = 0$ by semi-simplicity. So T is a bijection, and hence T is invertible. This proves the theorem.

<u>Theorem 6.4</u>. Let T be a multiplier of a commutative Banach algebra A as in Theorem 6.2, such that $\hat{T}$ vanishes at infinity. Then $T(A)$ is closed if and only if T is the product of an idempotent and an invertible multiplier.

<u>Proof</u>. The 'if' part follows from the first paragraph of this section. For the 'only if' part, let $K = \hat{T}^{-1}(0)$. If $T(A)$ is closed, K is open and closed and $\hat{T}$ is bounded away from zero on K^c. Since $\hat{T}$ vanishes at infinity, K^c is compact. By regularity, there exists $f \in A$, such that

$\hat{f} = \chi_{K^c}$, since K^c is a neighbourhood of itself. Since $\hat{f}$ has compact support disjoint from K = hull $(T(A))$, we have $f \in T(A)$. So there exists $g \in A$ such that $f = T(g)$, i.e. for any $m \in K^c$, $\hat{T}(m)\,\hat{g}(m) = 1$. Defining $T_1(f_1) = f_1 g$, we see that T_1 is a multiplier, such that T_1, T satisfy the hypothesis of Lemma 6.2. Therefore T is the product of an idempotent and an invertible multiplier. This completes the proof.

These results enable us to prove the following known theorem [26] on isometric multipliers of Banach algebras satisfying the hypothesis of Theorem 6.2.

<u>Theorem 6.5</u>. If T is an isometric multiplier of a commutative Banach algebra A as in Theorem 6.2, then T is surjective, and for all $m \in \mathcal{M}$, $|\hat{T}(m)| = 1$.

<u>Proof</u>. Let T be an isometric multiplier of A. Then $T^2(A)$ is closed. Thus by Theorem 6.2, $T = E\bar{T}'$, where E is an idempotent and $\bar{T}'$ is an invertible multiplier. Let $K = \hat{E}^{-1}(0)$. Then K is open and closed. If K is nonempty, choose $f \in A$, such that $f \neq 0$, and $\hat{f}$ is supported in K. Then $\widehat{T'(f)}$ is supported in K and hence $\widehat{T(f)} = \hat{E}(\widehat{T'(f)}) = 0$. But this contradicts the fact that T is an isometry. Thus K is empty, and hence $\hat{E}(m) = 1$ for all $m \in \mathcal{M}$. Therefore $E = I$, and $T = \bar{T}'$ is invertible and hence surjective.

To prove the rest of the assertion, let T_1 be the inverse of T. Then T_1 is a multiplier of A, and for all $m \in M$, $\hat{T}_1(m) = (\hat{T}(m))^{-1}$. Hence, for all $m \in M$, $|\hat{T}(m)| \leq ||T|| = 1$, and $|\hat{T}(m)|^{-1} = |\hat{T}_1(m)| \leq ||T_1|| = 1$. Therefore, $|\hat{T}(m)| = 1$ for all $m \in M$. This completes the proof.

Remark. As has been already proved in [26], Theorem 6.5 is actually true for any regular, commutative, semisimple, Tauberian Banach algebra.

In view of Theorem 6.1, the following is now obvious.

Theorem 6.6. Let A be a Banach algebra as in Theorem 6.2, and let G be a locally compact abelian group. Let T be a multiplier of $L^1(G,A)$. Then $T^2(L^1(G,A))$ is closed if and only if T is the product of an idempotent and an invertible multiplier. Moreover, if T is isometric then T is surjective, and $\hat{T}$ is of modulus one.

6.3 Application to Segal algebras: A Banach subalgebra $(S, ||\cdot||_S)$ of $L^1(G)$ is said to be a Segal algebra if S is a translation invariant dense subalgebra of $L^1(G)$, such that for every $f \in S$, the mapping $t \to f_t$ of G into S is continuous, and $||f_t||_S = ||f||_S$, for every $t \in G$. It follows from the definitions, that there exists some constant $\alpha > 0$, such that $||f||_1 \leq \alpha\, ||f||_S$, and usually this constant is assumed to be 1. From the definitions, it is easy to prove that S is an ideal in $L^1(G)$ and $||f * g||_S \leq ||f||_1 ||g||_S$ for any $f \in L^1(G)$ and $g \in S$. Also, the

maximal-ideal space of any Segal algebra is nothing but Γ, the dual of G, and any Segal algebra is semisimple, regular and Tauberian. Examples and discussion of these and other properties of Segal algebra, can be found in [29].
We now prove,

Proposition 6.1. Any Segal algebra S on a locally compact abelian group G satisfies the condition $*$.

Proof: Let V be an arbitrary neighbourhood of an arbitrary element γ of Γ. Choose $f \in S$, such that $\hat{f}_1(\gamma) = 1$, and $||f_1||_S \leq 2/||\gamma||_{S^*}$. Choose $f_2 \in L^1(G)$, such that $\hat{f}_2(\gamma) = 1$, $||f_2||_1 = 1$ and $\hat{f}_2$ is supported in V. This is possible by 2.6.1 of [34]. Let $f = f_2 * f_1$. Then $f \in S$, $\hat{f}(\gamma) = 1$, $||f||_S \leq ||f_2||_1 \, ||f_1||_S \leq 2/||\gamma||_{S^*}$, and $\hat{f}$ is supported in V. This proves that S satisfies the condition $*$ with $K = 2$.

In view of this, the results of §6.2 are applicable to Segal algebras, and we get [3],

Theorem 6.7. Let T be a multiplier of a Segal algebra S on G. Then $T^2(S)$ is closed if and only if T is the product of an idempotent and an invertible multiplier. Moreover, if T is an isometric multiplier, then T is surjective and $\hat{T}$ is of unit modulus.

These results can be used in a simple manner to prove the following known result on Segal algebras (see Proposition 2.2 of [21]).

Theorem 6.8. If G is non-compact, then the only compact multiplier of S is the zero operator.

Proof. Let T be a compact multiplier of a Segal algebra S on a noncompact group G. Let us take any complex number $\lambda \neq 0$. Then the range of $T-\lambda I$, where I is the identity operator, is closed by Theorem 4.23 of [35]. Since $T-\lambda I$ is also a multiplier, it follows from Lemma 6.1, that $K = \{\gamma \in \Gamma : \hat{T}(\gamma) = \lambda\} = \text{hull}\,[(T-\lambda I)(S)]$ is open and closed. Let $k(K^c) = \{f \in S : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin K\}$. Consider T', the restriction of T to $k(K^c)$. Obviously $T' = \lambda I$ on $k(K^c)$. Now T' is compact, and therefore $k(K^c)$ is finite dimensional. Let its dimension be n. Let K be nonempty if possible. Since G is noncompact, Γ is nondiscrete. Hence any nonempty open set in Γ has infinite number of points. Therefore we can find $(n+1)$ points $x_1, x_2, \ldots, x_{n+1}$ all belonging to K, and compact neighbourhoods $V_1, V_2, \ldots, V_{n+1}$ of $x_1, x_2, \ldots, x_{n+1}$ respectively, such that $V_i \cap V_j = \phi$ for $i \neq j$, and $V_i \subset K$ for $i = 1,2,\ldots,n+1$. Choose $f_i \in S$ for $i=1,2,\ldots,n+1$, such that $\hat{f}_i(x_i) = 1$, and support of $\hat{f}_i \subset V_i$. Obviously $\{f_i\}_{i=1}^{n+1}$ forms a linearly independent set of $k(K^c)$. But this contradicts the fact that the dimension of $k(K^c)$ is n. Hence K is empty, and therefore $\hat{T}(\gamma) \neq \lambda$ for all $\gamma \in \Gamma$. Since λ is an arbitrary nonzero complex number, we conclude that $\hat{T}(\gamma) = 0$ for all $\gamma \in \Gamma$. Hence $T = 0$, and this completes the proof.

6.4 <u>Glicksberg's conjecture</u>: Glicksberg in [7] conjectured that for $\mu \in M(G)$, $\mu * L^1(G)$ is closed if and only if μ is the convolve of an idempotent and an invertible measure. As observed by Glicksberg, this is true if Γ, the dual of G is connected or if $\mu \in L^1(G)$. This also follows from Theorems 6.3 and 6.4. Glicksberg also noted that to prove the conjecture, it is enough to prove that if $\mu * L^1$ is closed then there exists an idempotent $\nu \in M(G)$, such that $\nu * L^1 = \mu * L^1$. This fact can be obtained as a consequence of Lemma 6.2 as follows.

<u>Lemma 6.3</u>. Let $\mu \in M(G)$ be such that $\mu * L^1(G)$ is closed. Suppose there exists $\nu \in M(G)$, such that ν is an idempotent and $\nu * L^1 = \mu * L^1$. Then μ is the convolve of an idempotent and an invertible measure.

<u>Proof</u>. Take any $\gamma \in \Gamma$, such that $\hat{\mu}(\gamma) \neq 0$. Taking any $f \in L^1(G)$, such that $\hat{f}(\gamma) \neq 0$, we see that, there exists $g = \mu * f \in \mu * L^1$, such that $\hat{g}(\gamma) \neq 0$. Since $g \in \nu * L^1$, there exists $f_1 \in L^1(G)$, such that $\hat{f}_1(\gamma)\, \hat{\nu}(\gamma) = \hat{g}(\gamma) \neq 0$. This means that $\hat{\nu}(\gamma) \neq 0$. Similarly, we can show that if $\hat{\nu}(\gamma) \neq 0$, then $\hat{\mu}(\gamma) \neq 0$. Hence $\hat{\mu}^{-1}(0) = \hat{\nu}^{-1}(0) = K$ (say). Since ν is an idempotent, $\hat{\nu} = \chi_{K^c}$, and we get $\mu * \nu = \mu$. Since $\mu * L^1$ is closed, following the same arguments as in the proof of Theorem 6.2, we get $\mu * L^1 =$

$\nu * L^1 = k(K) = \{ f \in L^1 : \hat{f}=0 \text{ on } K \}$. Since $\mu*(\nu * L^1) = (\mu*\nu) * L^1 = \mu * L^1$, we see that $\mu * (k(K))=k(K)$. It is easy to see from the uniqueness of Fourier transform, that the map T given by $T(f) = \mu * f$, is an injection on $k(K)$. Hence T is a bijection of $k(K)$ onto $k(K)$. Let T_0 be the inverse of this map. Then T_0 is a bounded linear map from $k(K)$ onto $k(K)$. Consider the map T_1 from L^1 into L^1 given by $T_1(f) = T_0(\nu * f)$. It is easy to see that T_1 is bounded linear and

$$\widehat{T_1(f)}(\gamma)=\begin{cases} (\hat{\mu}(\gamma))^{-1}\hat{f}(\gamma) = (\hat{T}(\gamma))^{-1}\hat{f}(\gamma) & \text{on } K^c \\ 0 & \text{on } K \end{cases}$$

Hence T_1 is a multiplier of $L^1(G)$ and T_1 and T satisfy the hypothesis of Lemma 6.2. Therefore T is the product of an idempotent and an invertible multiplier, i.e. μ is the convolve of an idempotent and an invertible measure. This proves the lemma.

We shall now go one step forward towards the proof of Glickberg's conjecture by proving,

<u>Theorem 6.9</u>. Let $\mu \in M(G)$ such that $\mu * L^1$ is closed. If there exists a bounded projection P of $L^1(G)$ onto $\mu * L^1$, then μ is the convolve of an idempotent and an invertible measure.

Before proving this theorem, we note that this result is stronger than Lemma 6.3. In Lemma 6.3, one assumes the

existence of a measure ν with certain properties which is equivalent to the existence of an invariant bounded projection, i.e. a bounded projection which is also a multiplier. For compact groups, the existence of a bounded projection onto a closed translation invariant subspace, implies the existence of an invariant projection onto the same subspace (see Theorem 5.18 of [35]). However the following example due to Rosenthal (page 20 of [33]) shows that this is not true for noncompact groups.

Let $G = R$, the group of real numbers. The dual of R is R itself. Let $I = \{ f \in L^1(R) : \hat{f}(n) = 0,\ n \text{ integer} \}$. I is a closed translation invariant subspace and hull $I = Z$, the set of integers. Since R is connected, there cannot be any idempotent measure $\nu \in M(R)$, such that $\nu * L^1(R) = I$. However, we can construct or bounded projection of $L^1(G)$ onto I as follows. Let $f \in L^1(R)$. Then

$$\int_0^{2\pi} \Big(\sum_{n=-\infty}^{\infty} |f(x+2\pi n)| \Big)\, dx = \sum_{n=-\infty}^{\infty} \int_0^{\infty} |f(x+2\pi n)|\, dx$$

$$= ||f||_1 < \infty.$$

Therefore, $\sum_{n=-\infty}^{\infty} |f(x+2\pi n)| < \infty$ for almost all x in $[0,2\pi)$. So we can define a map U from $L^1(R)$ into $L^1(R)$ as follows,

$$Uf(x) = \begin{cases} \sum_{n=-\infty}^{\infty} f(x+2\pi n) & \text{for } x \in [0,2\pi) \\ 0 & \text{for } x \notin [0,2\pi). \end{cases}$$

It is easy to see that U is a bounded linear operator and that the map $f \to f - Uf$, is a bounded projection of $L^1(R)$ onto I.

We now prove two lemmas. The first lemma is of a trivial nature.

<u>Lemma 6.4</u>. Let (S, Σ, μ) be a measure space with μ a probability measure. Let H be a closed convex subset of the complex plane C, and let f be an integrable function on S with $f(S) \subset H$. Then $\int f d\mu \in H$.

<u>Proof</u>. Take any $Z_0 \notin H$. Choose $\varepsilon > 0$ such that $D(Z_0, \varepsilon) \cap H$ is empty, where $D(Z_0, \varepsilon)$ is the closed disk of radius ε around Z_0. By separation theorem, there exists a linear functional on C which strongly separates $D(Z_0, \varepsilon)$ and H. This means that there exists a real number α and a θ with $0 \leq \theta \leq 2\pi$, such that $\text{Re}(e^{i\theta} Z_0) < \alpha$ and $\text{Re}(e^{i\theta} Z) > \alpha$ for any $Z \in H$. Hence for any $s \in S$, $\text{Re}(e^{i\theta} f(s)) > \alpha$. Therefore,

$$\begin{aligned} \text{Re}(e^{i\theta} \textstyle\int f d\mu) - \alpha &= \int \text{Re}(e^{i\theta} f) d\mu - \alpha \int d\mu \\ &= \int \left[\text{Re}(e^{i\theta} f(s)) - \alpha \right] d\mu(s) > 0. \end{aligned}$$

Hence $\text{Re}(e^{i\theta} \int f d\mu) > \alpha$. Therefore $e^{i\theta} \int f d\mu \neq e^{i\theta} Z_0$, i.e. $\int f d\mu \neq Z_0$. This is true for any $Z_0 \notin H$, and hence $\int f d\mu \in H$. This proves the lemma.

Lemma 6.5. Let $L \subset M(G)$, such that for any $\nu \in L$, $\hat{\nu}(U) \subset H$, where U is a neighbourhood of some $\gamma_0 \in \Gamma$, and H is a closed convex subset of the complex plane. Let μ be a weak*-limit point of L. Then $\hat{\mu}(\gamma_0) \in H$.

Proof. Suppose $\hat{\mu}(\gamma_0) = Z_0 \notin H$. Choose $\varepsilon > 0$ such that $D(Z_0,\varepsilon) \cap H$ is empty. Choose a compact neighbourhood V of γ_0, such that $V \subset U$, and for any $\gamma \in V$, $\hat{\mu}(\gamma) \in D(Z_0,\varepsilon)$. Choose $k \in L^1(G)$, such that $\hat{k}$ is supported in V, $\hat{k} \geq 0$ on Γ and $\int_\Gamma \hat{k}(\gamma)d\gamma = \int_V \hat{k}(\gamma)d\gamma = 1$. Since $\hat{k} \in L^1(\Gamma)$, by inversion theorem, $k(x) = \int_\Gamma (x,\gamma)\, \hat{k}(\gamma)d\gamma$, and hence $k \in C_0(G)$. Now for any $\lambda \in M(G)$,

$$\begin{aligned} \langle k,\lambda \rangle &= \int_G k(-x)\, d\lambda(x) \\ &= \int_G \left(\int_\Gamma (-x,\gamma)\, \hat{k}(\gamma)d\gamma\right) d\lambda(x) \\ &= \int_\Gamma \hat{k}(\gamma)d\gamma \left(\int_G (-x,\gamma)\, d\lambda(x)\right) \\ &= \int_V \hat{\lambda}(\gamma)\, \hat{k}(\gamma)d\gamma \end{aligned}$$

Since $\hat{k}d\gamma$ gives a probability measure on V, and $\hat{\nu}(V) \subset \hat{\nu}(U) \subset H$ for any $\nu \in L$, we can use Lemma 6.4 to conclude that for any $\nu \in L$, $\langle k,\nu \rangle = \int_V \hat{\nu}(\gamma)\, \hat{k}(\gamma)d\gamma \in H$. Since μ is a weak*-limit point of L, and H is closed, we have $\langle k,\mu \rangle \in H$. On the other hand, for any $\gamma \in V$, $|\hat{\mu}(\gamma) - Z_0| \leq \varepsilon$. Hence,

$$|\langle k,\mu\rangle - Z_0| = \left| \int_V k(\gamma)\, \mu(\gamma) d\gamma - \int_V Z_0 \hat{k}(\gamma) d\gamma \right|$$

$$\leq \int_V |\mu(\gamma) - Z_0| k(\gamma) d\gamma$$

$$\leq \varepsilon \int_V k(\gamma) d\gamma = \varepsilon$$

Therefore, $\langle k,\mu\rangle \in D(Z_0,\varepsilon)$ Since $D(Z_0,\varepsilon)$ is disjoint from H, this is a contradiction This contradiction proves the lemma

Proof of Theorem 6 9 Let $K = \mu^{-1}(0)$ Since μ L^1 is closed, by Lemma 6 1 we conclude that K is open and closed It follows that $\mu * L^1(G) = I = k(K) = \{f \in L^1 \; f = 0 \text{ on } K\}$ Let $g \in L^1(G)$ be a weak*-limit point of I Since K is open, taking any $\gamma \in K$ and putting K for U, $\{0\}$ for H and I for L in Lemma 6 5, we see that $g(\gamma) = 0$ Therefore $g \in I$, and hence I is a weak*-closed ideal in $L^1(G)$ Also there exists a bounded projection of $L^1(G)$ onto I Hence by Theorem 4 of [24], we see that $I = \nu * L^1(G)$ for some idempotent $\nu \in L^1(G)$ Therefore by Lemma 6 3, it follows that μ is the convolve of an idempotent and an invertible measure This completes the proof

B Host and F Parreau [16] have proved Glicksberg's conjecture by using the concept of critical points discussed in [39] In view of our generalisations of Glicksberg's results, we can ask the question whether Glicksberg's conjecture

is true for algebras other than $L^1(G)$. For example, it will be interesting to see whether the conjecture is true for Segal algebras. To treat this question, we need methods different from those of [16], where the special nature of $L^1(G)$ has been used to get the result for $L^1(G)$. In view of the considerable efforts required to prove the conjecture for $L^1(G)$, we can safely say that the general case will not be at all easy to resolve, and it gives us a challenging open problem.

REFERENCES

1. N. Dinculeanu, 'Vector Measures', Pergamon Press, Oxford, 1967.

2. N. Dunford and B. Pettis, 'Linear operations on summable functions', Trans. Amer. Math. Soc., 47(1940), 323-392.

3. M. Dutta and U.B. Tewari, 'On multipliers of Segal algebra', Proc. Amer. Math. Soc. (to be published).

4. G.I. Gaudry, 'Topics in Harmonic Analysis', Yale Lecture Notes (mimeographed), 1969.

5. B.R. Gelbaum, 'Tensor products of Banach Algebras', Canad. J. of Math., 11(1959) 297-310.

6. B.R. Gelbaum, 'Tensor products and related questions', Trans. Amer. Math. Soc., 103(1962) 525-548.

7. I. Glicksberg, 'When is $\mu * L_1$ closed?', Trans. Amer. Math. Soc., 160(1971) 419-425.

8. A. Grothendieck, 'Produits tensoriels topologiques et espaces nucleaires', Mem. Amer. Math. Soc. No. 16, 1955.

9. P.R. Halmos, 'Measure Theory', D. Van Nostrand Co., 1950.

10. A. Hausner, 'Group algebras of vector-valued functions', Bull. Amer. Math. Soc. 62 (1956) 383 (abstract 493).

11. A Hausner, 'The Tauberian theorem for group algebra of vector-valued functions', Pacific J. Math., 7(1957), 1603-1610.

12. A. Hausner, 'On generalised group algebras', Proc. Amer. Math. Soc., 10(1959) 1-10.

13. E. Hewitt and K.A. Ross, 'Abstract Harmonic Analysis', Vol. I, Springer Verlag, Berlin-Heidelberg-New York, 1963.

14. E. Hewitt and K.A. Ross, 'Abstract Harmonic Analysis', Vol. II, Springer Verlag, Berlin-Heidellberg-New York, 1970.

15. E. Hille and R.S. Phillips, 'Functional Analysis and Semigroups', Amer. Math. Soc. Colloq. Pub., Vol. 31, New York, 1957.

16. B. Host and F. Parreau, 'Sur la fermeture de $\mu * L^1(G)$', C.R. Acad. Sc. Paris, Serie A, 285(1977) 15-17.

17. J.E. Huneycutt Jr., 'Products and convolutions of vector-valued set functions', Studia Mathematica, 41(1972) 101-129.

18. G.P. Johnson, 'Spaces of functions with values in a Banach algebra', Trans. Amer. Math. Soc., 92(1959) 411-429.

19. S. Kakutani, 'Concrete Representations of abstract(L)-spaces and the mean ergodic theorem', Annals of Math. (2) 42(1941) 523-537.

20. J.L. Kelley, 'General Topology', D.Van Nostrand Co., 1955.

21. H.E. Krogstad, 'Multipliers of Segal algebras', Math. Scand., 38(1976) 285-303.

22. J.G. Lamadrid, 'Measures and Tensors', Trans. Amer. Math. Soc., 114(1965) 98-121.

23. R. Larsen, 'An introduction to the theory of multipliers', Springer Verlag, Berlin-Heidellberg-New York (1971).

24. T. Liu et al, 'Projections and approximate identities for ideals in group algebras', Trans. Amer. Math. Soc. 175(1973) 469-482.

25. L.H. Loomis, 'An introduction to Abstract Harmonic Analysis', D. Van Nostrand Co., 1953.

26. K. Parthasarathy, 'Segal algebras-some explorations', Ph.D. Thesis, Indian Institute of Technology, Kanpur, 1977.

27. R. S. Phillips, 'On weakly compact subsets of a Banach space', Amer. J. of Math., 65(1943) 108-136.

28. H. Reiter, 'Classical Harmonic Analysis and locally compact groups', Oxford Univ. Press, 1968.

29. H. Reiter, 'L^1-algebra and Segal algebras', Lecture Notes in Math. Vol. 231, Springer Verlag, Berlin-Heidelberg-New York, 1971.

30. M. Rieffel, 'A characterisation of commutative group algebras and measure algebras', Trans. Amer. Math. Soc., 116(65), 32-64.

31. M. Rieffel, 'Induced Banach representations of Banach algebras and locally compact groups', J. of Functional Analysis, 1(1967) 443-491.

32. M. Rieffel, 'Multipliers and tensor products of L^p-spaces of locally compact groups', Studia Mathematica, 33(1969) 71-82.

33. H.P. Rosenthal, 'Projections onto translation invariant subspaces of $L^p(G)$, Mem. Am. Math. Soc. No. 63, 1966.

34. W. Rudin, 'Fourier Analysis on groups', Interscience Tracts in Pure and Appl. Math. no. 12, Interscience, New York, 1962.

35. W. Rudin, 'Functional Analysis', Mc-Graw Hill Inc., 1973.

36. H.H. Schaefer, 'Topological Vector Spaces', Mc-Millan, New York, 1966.

37. R. Schatten, 'A theory of cross-spaces', Annals of Mathematics Studies no. 26, Princeton Univ. Press, 1950.

38. I. Singer, 'Linear functionals on spaces of continuous mappings of a compact Hausdorff space into a Banach space', Rev. Math. Pures. Appl. 2(1957) 309-315.

39. J.L. Taylor, 'Measure Algebras' Regional Conference Series in Math. no. 16, Conference Board of Mathematical Sciences, 1973.

40. J. Tomiyama, 'Tensor products of commutative Banach algebras', Tohuku Math. J. 12(1960) 147-154.

41. J.J. Uhl Jr., 'The range of a vector-valued measure', Proc. Amer. Math. Soc. 23(1969) 158-163.

42. A. Weil, 'L'integration dans les groupes topologiques et ses applications', Hermann, Paris, 1951.

43. A.J. White, 'Ordered Banach algebras', Journal London Math. Soc., Ser. 2, 11 (1975) 175-178.